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Tropical Geometry, Mathematical Morphology and Weighted Lattices

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Abstract. Mathematical Morphology and Tropical Geometry share the same max/min-plus scalar arithmetic and matrix algebra. In this paper we summarize their common ideas and algebraic structure, generalize and extend both of them using weighted lattices and a max-$\star$ algebra with an arbitrary binary operation $\star$ that distributes over max, and outline applications to geometry, image analysis, and optimization. Further, we outline the optimal solution of max-$\star$ equations using weighted lattice adjunctions, and apply it to optimal regression for fitting max-$\star$ tropical curves on arbitrary data.

Keywords: Tropical Geometry · Morphology · Weighted lattices

1 Introduction

Max-plus convolutions have appeared in morphological image analysis, convex analysis and optimization [3,13,17,19,23,25], and nonlinear dynamical systems [2,21]. Max-plus or its dual min-plus arithmetic and corresponding matrix algebra have been used in operations research and scheduling [10]; discrete event control systems, max-plus control and optimization [1,2,6,8]; idempotent mathematics [16]. Max-plus arithmetic is an idempotent semiring; as such it is covered by the theory of dioids [12]. The dual min-plus has been called 'tropical semiring' and has been used in finite automata [15] and tropical geometry [18].

Max and min operations (or more generally supremum and infimum) form the algebra of lattices, which has been used to generalize Euclidean morphology based on Minkowski set operations and their extensions to functions via level sets to morphology on complete lattices [13,14,24]. The scalar arithmetic of morphology on functions has been mainly flat; a few exceptions include max-plus convolutions in [25] and related operations of the image algebra in [22]. Such non-flat morphological operations and their generalizations to a max-$\star$ algebra have been systematized using the theory of weighted lattices [20,21]. This connects morphology with max-plus algebra and tropical geometry.

Mathematical Morphology and Tropical Geometry share the same max/min-plus scalar arithmetic and max/min-plus matrix algebra. In this paper we summarize their common ideas and algebraic structure, extend both of them using

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weighted lattices, and outline applications to geometry, image analysis and optimization. We begin with some elementary concepts from morphological operators and tropical geometry. Then, we extend the underlying max-plus algebra to a max-$\star$ algebra where matrix operations and signal convolutions are generalized using a (max,$\star$) arithmetic with an arbitrary binary operation $\star$ that distributes over max. This theory is based on complete weighted lattices and allows for both finite- and infinite-dimensional spaces. Finally, we outline the optimal solution of systems of max-$\star$ equations using weighted lattice adjunctions and projections, and apply it to optimal regression for fitting max-$\star$ tropical curves on data.

2 Background: Morphology on Flat Lattices

We view images, signals and vectors as elements of complete lattices $(\mathcal{L}, \vee, \wedge)$, like the set $\text{Fun}(E, \overline{\mathbb{R}})$ of functions with domain E and values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, and consider operators on $\mathcal{L}$, i.e., mappings from $\mathcal{L}$ to itself.

Monotone Operators: A lattice operator ψ is called *increasing* (resp. *decreasing*) if it is order preserving (resp. inverting). Examples of increasing operators are the lattice homomorphisms which preserve suprema and infima. If a lattice homomorphism is also a bijection, then it becomes an automorphism. Four fundamental types of increasing operators are: *dilations* δ and *erosions* ε that satisfy respectively $\delta(\bigvee_i X_i) = \bigvee_i \delta(X_i)$ and $\varepsilon(\bigwedge_i X_i) = \bigwedge_i \varepsilon(X_i)$ over arbitrary (possibly infinite) collections; *openings* α that are increasing, idempotent and antiextensive; *closings* β that are increasing, idempotent and extensive. Openings and closings are *lattice projections*. Examples of decreasing operators are the dual homomorphisms, which interchange suprema with infima. A lattice dual automorphism is a bijection that interchanges suprema with infima. A *negation* ν is a dual automorphism that is also involutive.

Residuation and Adjunctions: An increasing operator ψ on a complete lattice $\mathcal{L}$ is called **residuated** [5] if there exists an increasing operator $\psi^\sharp$ such that

$$\psi\psi^\sharp \leq \mathbf{id} \leq \psi^\sharp\psi \tag{1}$$

$\psi^\sharp$ is called the **residual** of ψ and is the closest to being an inverse of ψ. Specifically, the residuation pair $(\psi, \psi^\sharp)$ can solve inverse problems of the type $\psi(X) = Y$ either exactly since $\hat{X} = \psi^\sharp(Y)$ is the greatest solution of $\psi(X) = Y$ if a solution exists, or approximately since $\hat{X}$ is the greatest *subsolution* in the sense that $\hat{X} = \bigvee\{X : \psi(X) \leq Y\}$. On complete lattices an increasing operator ψ is residuated (resp. a residual $\psi^\sharp$) if and only if it is a dilation (resp. erosion). The residuation theory has been used for solving inverse problems in matrix algebra [2,9,10] over the max-plus or other idempotent semirings.

Dilations and erosions on a complete lattice $\mathcal{L}$ come in pairs (δ, ε) of operators; such a pair is called **adjunction** on $\mathcal{L}$ if

$$\delta(X) \leq Y \iff X \leq \varepsilon(Y) \quad \forall X, Y \in \mathcal{L} \tag{2}$$

The double inequality (2) is equivalent to the inequality (1) satisfied by a residuation pair of increasing operators if we identify the residuated map ψ with δ and its residual $\psi^\sharp$ with ε. There is a one-to-one correspondence between the two operators of an adjunction; e.g., given a dilation δ, there is a unique erosion $\varepsilon(Y) = \bigvee\{X \in \mathcal{L} : \delta(X) \leq Y\}$ such that (δ, ε) is adjunction. An adjunction (δ, ε) automatically yields two lattice projections, an opening $\alpha = \delta\varepsilon$ and a closing $\beta = \varepsilon\delta$, such that $\delta\varepsilon \leq \mathbf{id} \leq \varepsilon\delta$. There are also other types of lattice projections studied in [9].

3 Weighted Lattices

Lattice-Ordered Monoids and Clodum: A lattice $(\mathcal{K}, \vee, \wedge)$ is often endowed with a third binary operation, called symbolically the 'multiplication' $\star$, under which $(\mathcal{K}, \star)$ is a group, or a monoid, or just a semigroup [4]. Consider now an algebra $(\mathcal{K}, \vee, \wedge, \star, \star')$ with four binary operations, which we call a *lattice-ordered double monoid*, where $(\mathcal{K}, \vee, \wedge)$ is a lattice, $(\mathcal{K}, \star)$ is a monoid whose 'multiplication' $\star$ distributes over $\vee$, and $(\mathcal{K}, \star')$ is a monoid whose 'multiplication' $\star'$ distributes over $\wedge$. These distributivities imply that both $\star$ and $\star'$ are increasing. To the above definitions we add the word *complete* if $\mathcal{K}$ is a complete lattice and the distributivities involved are infinite. We call the resulting algebra a *complete lattice-ordered double monoid*, in short *clodum* [19–21]. Previous works on minimax or max-plus algebra have used alternative names for structures similar to the above definitions which emphasize semigroups and semirings instead of lattices [2,10,12]; see [21] for similarities and differences. We precisely define an algebraic structure $(\mathcal{K}, \vee, \wedge, \star, \star')$ to be a **clodum** if:

(C1) $(\mathcal{K}, \vee, \wedge)$ is a complete distributive lattice. Thus, it contains its least $\bot := \bigwedge \mathcal{K}$ and greatest element $\top := \bigvee \mathcal{K}$. The supremum $\vee$ (resp. infimum $\wedge$) plays the role of a generalized *'addition'* (resp. *'dual addition'*).

(C2) $(\mathcal{K}, \star)$ is a monoid whose operation $\star$ plays the role of a generalized *'multiplication'* with identity ('unit') element e and is a dilation.

(C3) $(\mathcal{K}, \star')$ is a monoid with identity e' whose operation $\star'$ plays the role of a generalized *'dual multiplication'* and is an erosion.

Remarks: (i) As a lattice, $\mathcal{K}$ is not necessarily infinitely distributive, although in this paper all our examples will be such.
(ii) The clodum 'multiplications' $\star$ and $\star'$ do not have to be commutative.
(iii) The least (greatest) element $\bot$ ($\top$) of $\mathcal{K}$ is both the 'zero' element for the 'addition' $\vee$ ($\wedge$) and an absorbing null for the 'multiplication' $\star$ ($\star'$).
(iv) We avoid degenerate cases by assuming that $\vee \neq \star$ and $\wedge \neq \star'$. However, $\star$ may be the same as $\star'$, in which case we have a self-dual 'multiplication'.
(v) A clodum is called *self-conjugate* if it has a lattice negation $a \mapsto a^*$.

If $\star = \star'$ over $G = \mathcal{K} \setminus \{\bot, \top\}$ where $(G, \star)$ is a group and $(G, \vee, \wedge)$ a conditionally complete lattice, then the clodum $\mathcal{K}$ becomes a richer structure which we call a *complete lattice-ordered group*, in short **clog**. In any clog the distributivity between $\vee$ and $\wedge$ is of the infinite type and the 'multiplications' $\star$ and $\star'$ are commutative. Then, for each $a \in G$ there exists its 'multiplicative inverse' a^{-1} such that $a \star a^{-1} = e$. Further, the 'multiplication' $\star$ and its self-dual $\star'$ (which coincide over G) can be extended over the whole $\mathcal{K}$ by involving the null elements. A clog becomes self-conjugate by setting $a^* = a^{-1}$ if $\perp < a < \top$, $\top^* = \perp$, and $\perp^* = \top$. In a clog $\mathcal{K}$ the $\star$ and $\star'$ coincide in all cases with only one exception: the combination of the least and greatest elements; thus, we can denote the clog algebra as $(\mathcal{K}, \vee, \wedge, \star)$.

Example 1. (a) *Max-plus* clog $(\overline{\mathbb{R}}, \vee, \wedge, +, +')$: $\vee/\wedge$ denote the standard sup/inf on $\overline{\mathbb{R}}$, $+$ is the standard addition on $\overline{\mathbb{R}}$ playing the role of a 'multiplication' $\star$ with $+'$ being the 'dual multiplication' $\star'$; the operations $+$ and $+'$ are identical for finite reals, but $a + (-\infty) = -\infty$ and $a +' (+\infty) = +\infty$ for all $a \in \overline{\mathbb{R}}$. The identities are $e = e' = 0$, the nulls are $\perp = -\infty$ and $\top = +\infty$, and the conjugation mapping is $a^* = -a$.
(b) *Max-times* clog $([0, +\infty], \vee, \wedge, \times, \times')$: The identities are $e = e' = 1$, the nulls are $\perp = 0$ and $\top = +\infty$, and the conjugation mapping is $a^* = 1/a$.
(c) *Max-min* clodum $([0, 1], \vee, \wedge, \min, \max)$: As 'multiplications' we have $\star = \min$ and $\star' = \max$. The identities and nulls are $e' = \perp = 0$, $e = \top = 1$. A possible conjugation mapping is $a^* = 1 - a$. Additional clodums that are not clogs are discussed in [19,21] using more general fuzzy intersections and unions.
(d) *Matrix* max-$\star$ clodum: $(\mathcal{K}^{n\times n}, \vee, \wedge, \boxast, \boxast')$ where $\mathcal{K}^{n\times n}$ is the set of $n \times n$ matrices with entries from a clodum $\mathcal{K}$, $\vee/\wedge$ denote here elementwise matrix sup/inf, and $\boxast$, $\boxast'$ denote max-$\star$ and min-$\star'$ matrix 'multiplications':

$$\mathbf{C} = \mathbf{A} \boxast \mathbf{B} = [c_{ij}],\ c_{ij} = \bigvee_{k=1}^{n} a_{ik} \star b_{kj}\ ,\ \mathbf{D} = \mathbf{A} \boxast' \mathbf{B} = [d_{ij}],\ d_{ij} = \bigwedge_{k=1}^{n} a_{ik} \star' b_{kj}$$

This is a clodum with non-commutative 'multiplications'.

Complete Weighted Lattices: Consider a nonempty collection $\mathcal{W}$ of mathematical objects, which will be our space; examples of such objects include the vectors in $\overline{\mathbb{R}}^n$ or signals in $\mathrm{Fun}(E, \overline{\mathbb{R}})$. Also, consider a clodum $(\mathcal{K}, \vee, \wedge, \star, \star')$ of **scalars** with *commutative* operations $\star, \star'$ and $\mathcal{K} \subseteq \overline{\mathbb{R}}$. We define *two internal operations* among vectors/signals X, Y in $\mathcal{W}$: their supremum $X \vee Y : \mathcal{W}^2 \to \mathcal{W}$ and their infimum $X \wedge Y : \mathcal{W}^2 \to \mathcal{W}$, which we denote using the same supremum symbol ($\vee$) and infimum symbol ($\wedge$) as in the clodum, hoping that the differences will be clear to the reader from the context. Further, we define *two external operations* among any vector/signal X in $\mathcal{W}$ and any scalar c in $\mathcal{K}$: a 'scalar multiplication' $c \star X : (\mathcal{K}, \mathcal{W}) \to \mathcal{W}$ and a 'scalar dual multiplication' $c \star' X : (\mathcal{K}, \mathcal{W}) \to \mathcal{W}$, again by using the same symbols as in the clodum. Now, we define $\mathcal{W}$ to be a **weighted lattice** space over the clodum $\mathcal{K}$ if for all $X, Y, Z \in \mathcal{W}$ and $a, b \in \mathcal{K}$ all the axioms of Table 3 in [21] hold. These axioms bear a striking similarity with those of a linear space. One difference is that the vector/signal addition ($+$) of linear spaces is now replaced by two dual superpositions, the lattice supremum ($\vee$) and infimum ($\wedge$); further, the scalar multiplication ($\times$) of linear spaces is now replaced by two operations $\star$ and $\star'$ which are dual to each other. Only one major property of the linear spaces is missing from the weighted lattices: the existence of 'additive inverses'. We define the weighted lattice $\mathcal{W}$ to be a **complete weighted lattice (CWL)** space if (i) $\mathcal{W}$ is closed under any (possibly infinite) suprema and infima, and (ii) the distributivity laws between the scalar operations $\star$ ($\star'$) and the supremum (infimum) are of the infinite type. Note that, a commutative clodum is a complete weighted lattice over itself.

4 Vector and Signal Operators on Weighted Lattices

We focus on CWLs whose underlying set is a *space* $\mathcal{W} = \mathrm{Fun}(E, \mathcal{K})$ of *functions* $f : E \to \mathcal{K}$ with values from a clodum $(\mathcal{K}, \vee, \wedge, \star, \star')$ of scalars as in Examples 1(a), (b), (c). Such functions include n-dimensional vectors if $E = \{1, 2, ..., n\}$ or d-dimensional signals of continuous ($E = \mathbb{R}^d$) or discrete domain ($E = \mathbb{Z}^d$). Then, we extend *pointwise* the supremum, infimum and scalar multiplications of $\mathcal{K}$ to the functions: e.g., for $F, G \in \mathcal{W}$, $a \in \mathcal{K}$ and $x \in E$, we define $(F \vee G)(x) := F(x) \vee G(x)$ and $(a \star F)(x) := a \star F(x)$. Further, the scalar operations $\star$ and $\star'$, extended pointwise to functions, distribute over any suprema and infima, respectively. If the clodum $\mathcal{K}$ is self-conjugate, then we can extend the conjugation $(\cdot)^*$ to functions F pointwise: $F^*(x) \triangleq (F(x))^*$.

Elementary increasing operators on $\mathcal{W}$ are those that act as **vertical translations** (in short V-translations) of functions. Specifically, pointwise 'multiplications' of functions $F \in \mathcal{W}$ by scalars $a \in \mathcal{K}$ yield the *V-translations* τ_a and *dual V-translations* τ'_a, defined by $\tau_a(F)](x) := a \star F(x)$ and $\tau'_a(F)](x) := a \star' F(x)$. A function operator ψ on $\mathcal{W}$ is called **V-translation invariant** if it commutes with any V-translation τ, i.e., $\psi\tau = \tau\psi$. Similarly for dual translations.

Every function $F(x)$ admits a representation as a supremum of V-translated impulses placed at all points or as infimum of dual V-translated impulses:

$$F(x) = \bigvee_{y \in E} F(y) \star q_y(x) = \bigwedge_{y \in E} F(y) \star' q'_y(x) \tag{3}$$

where $q_y(x) = e$ at $x = y$ and $\perp$ else, whereas $q'_y(x) = e'$ at $x = y$ and $\top$ else. By using the V-translations and the representation of functions with impulses, we can build more complex increasing operators. We define operators δ as **dilation V-translation invariant (DVI)** and operators ε as **erosion V-translation invariant (EVI)** iff for any $c_i \in \mathcal{K},\ F_i \in \mathcal{W}$

$$\mathrm{DVI} : \delta(\bigvee_i c_i \star F_i) = \bigvee_i c_i \star \delta(F_i), \quad \mathrm{EVI} : \varepsilon(\bigwedge_i c_i \star' F_i) = \bigwedge_i c_i \star' \varepsilon(F_i) \tag{4}$$

The structure of a DVI or EVI operator's output is simplified if we express it via the operator's impulse responses. Given a dilation δ on $\mathcal{W}$, its **impulse response map** is the map $H : E \to \mathrm{Fun}(E, \mathcal{K})$ defined at each $y \in E$ as the output function $H(x, y)$ from δ when the input is the impulse $q_y(x)$. Dually, for an erosion operator ε we define its *dual impulse response map* H' via its outputs when excited by dual impulses: for $x, y \in E$

$$H(x,y) \triangleq \delta(q_y)(x), \quad H'(x,y) \triangleq \varepsilon(q'_y)(x) \tag{5}$$

Applying a DVI operator δ or an EVI operator ε to (3) and using the definitions in (5) proves the following unified representation.

Theorem 1. *(a) An operator δ on $\mathcal{W}$ is DVI iff its output can be expressed as*

$$\delta(F)(x) = \bigvee_{y \in E} H(x,y) \star F(y) \tag{6}$$

(b) An operator ε on $\mathcal{W}$ is EVI iff its output can be expressed as

$$\varepsilon(F)(x) = \bigwedge_{y \in E} H'(x,y) \star' F(y) \tag{7}$$

On signal spaces, the operations (6) and (7) are *shift-varying nonlinear convolutions.*

Weighted Lattice of Vectors: Consider now the nonlinear vector space $\mathcal{W} = \mathcal{K}^n$, equipped with the pointwise partial ordering $\mathbf{x} \leq \mathbf{y}$, supremum $\mathbf{x} \vee \mathbf{y} = [x_i \vee y_i]$ and infimum $\mathbf{x} \wedge \mathbf{y} = [x_i \wedge y_i]$ between any vectors $\mathbf{x}, \mathbf{y} \in \mathcal{W}$. Then, $(\mathcal{W}, \vee, \wedge, \star, \star')$ is a complete weighted lattice. Elementary increasing operators are the *vector V-translations* $\mathcal{T}_a(\mathbf{x}) = a \star \mathbf{x} = [a \star x_i]$ and their duals $\mathcal{T}'_a(\mathbf{x}) = a \star' \mathbf{x}$, which 'multiply' a scalar a with a vector $\mathbf{x}$ elementwise. A vector transformation on $\mathcal{W}$ is called (dual) V-translation invariant if it commutes with any vector (dual) V-translation. By defining as 'impulses' the impulse vectors $\mathbf{q}_j = [q_j(i)]$ and their duals $\mathbf{q}'_j = [q'_j(i)]$, where the index j signifies the position of the identity, each vector $\mathbf{x} = [x_1, ..., x_n]^T$ has a representation as a max of V-translated impulse vectors or as a min of V-translated dual impulse vectors. More complex examples of increasing operators on this vector space are the max-$\star$ and the min-$\star'$ 'multiplications' of a matrix $\mathbf{A}$ with an input vector $\mathbf{x}$,

$$\delta_{\mathbf{A}}(\mathbf{x}) \triangleq \mathbf{A} \boxast \mathbf{x}, \quad \varepsilon_{\mathbf{A}}(\mathbf{x}) \triangleq \mathbf{A} \boxast' \mathbf{x} \tag{8}$$

which are the prototypes of any vector transformation that obeys a sup-$\star$ or an inf-$\star'$ superposition.

Theorem 2. *(a) Any vector transformation on the complete weighted lattice $\mathcal{W} = \mathcal{K}^n$ is DVI iff it can be represented as a matrix-vector max-$\star$ product $\delta_{\mathbf{A}}(\mathbf{x}) = \mathbf{A} \boxast \mathbf{x}$ where $\mathbf{A} = [a_{ij}]$ with $a_{ij} = \{\delta(\mathbf{q}_j)\}_i$.*
(b) Any vector transformation on $\mathcal{K}^n$ is EVI iff it can be represented as a matrix-vector min-$\star'$ product $\varepsilon_{\mathbf{A}}(\mathbf{x}) = \mathbf{A} \boxast' \mathbf{x}$ where $\mathbf{A} = [a_{ij}]$ with $a_{ij} = \{\varepsilon(\mathbf{q}'_j)\}_i$.

Given a vector dilation $\delta(\mathbf{x}) = \mathbf{A} \boxast \mathbf{x}$, there corresponds a unique erosion ε so that (δ, ε) is a *vector adjunction* on $\mathcal{W}$, i.e. $\delta(\mathbf{x}) \leq \mathbf{y} \iff \mathbf{x} \leq \varepsilon(\mathbf{y})$. We can find the adjoint vector erosion by decomposing both vector operators based on scalar operators (η, ζ) that form a *scalar adjunction* on $\mathcal{K}$:

$$\eta(a,v) \leq w \Longleftrightarrow v \leq \zeta(a,w) \tag{9}$$

If we use as scalar 'multiplication' a commutative binary operation $\eta(a,v) = a \star v$ that is a dilation on $\mathcal{K}$, its scalar adjoint erosion becomes

$$\zeta(a,w) = \sup\{v \in \mathcal{K} : a \star v \leq w\} \tag{10}$$

which is a (possibly non-commutative) binary operation on $\mathcal{K}$. Then, the original vector dilation $\delta(\mathbf{x}) = \mathbf{A} \boxast \mathbf{x}$ is decomposed as

$$\{\delta(\mathbf{x})\}_i = \bigvee_j \eta(a_{ij}, x_j) = \bigvee_j a_{ij} \star x_j, \quad i = 1, ..., n \tag{11}$$

whereas its adjoint vector erosion is decomposed as

$$\{\varepsilon(\mathbf{y})\}_j = \bigwedge_i \zeta(a_{ij}, y_i), \quad j = 1, ..., n \tag{12}$$

The latter can be written as a min-ζ matrix-vector multiplication

$$\varepsilon(\mathbf{y}) = \mathbf{A}^T \Box'_\zeta \mathbf{y} \tag{13}$$

Further, if $\mathcal{K}$ is a *clog*, then $\zeta(a,w) = a^* \star' w$ and hence

$$\varepsilon(\mathbf{y}) = \mathbf{A}^* \boxast' \mathbf{y}, \tag{14}$$

where $\mathbf{A}^* = [a_{ji}{}^*]$ is the *adjoint* (i.e. conjugate transpose) of $\mathbf{A} = [a_{ij}]$.

Weighted Lattice of Signals: Consider the set $\mathcal{W} = \text{Fun}(E, \mathcal{K})$ of all signals $f : E \to \mathcal{K}$ with values from $\mathcal{K}$. The signal translations are the operators $\tau_{k,v}(f)(t) = f(t-k) \star v$ and their duals. A signal operator on $\mathcal{W}$ is called *(dual) translation invariant* iff it commutes with any such (dual) translation. This translation-invariance contains both a vertical translation and a horizontal translation (shift). Consider now operators Δ on $\mathcal{W}$ that are dilations and translation-invariant. Then Δ is both DVI in the sense of (4) and shift-invariant. We call such operators **dilation translation-invariant (DTI)** systems. Applying Δ to an input signal f decomposed as supremum of translated impulses yields its output as the sup-$\star$ convolution $\circledast$ of the input with the system's impulse response $h = \Delta(q)$, where $q(x) = e$ if $x = 0$ and $\perp$ else:

$$\Delta(f)(x) = (f \circledast h)(x) = \bigvee_{y \in E} f(y) \star h(x-y) \tag{15}$$

Conversely, every sup-$\star$ convolution is a DTI system. As done for the vector operators, we can also build signal operator pairs $(\Delta, \mathcal{E})$ that form adjunctions.

Given Δ we can find its adjoint $\mathcal{E}$ from scalar adjunctions (η, ζ). Thus, by (9) and (10), if $\eta(h, f) = h \star f$, the adjoint signal erosion becomes

$$\mathcal{E}(g)(y) = \bigwedge_{x \in E} \zeta[h(x-y), g(x)] \tag{16}$$

Further, if $\mathcal{K}$ is a clog, then

$$\mathcal{E}(g)(y) = \bigwedge_{x \in E} g(x) \star' h^*(x-y) \tag{17}$$

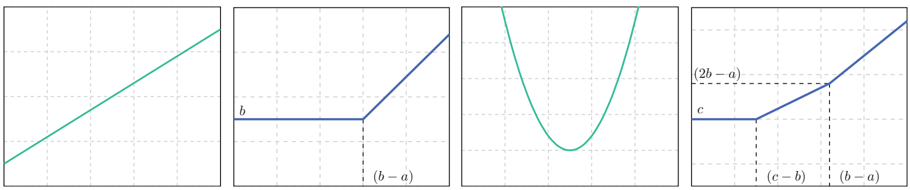
5 Tropical Geometry and CWL Generalizations

Tropical geometry [18] is an extension of analytic Euclidean geometry where the traditional arithmetic of the real field $(\mathbb{R}, +, \times)$ involved in the analytic expressions of geometric objects is replaced by the arithmetic of the min-plus tropical semiring $(\mathbb{R}_{\min}, \wedge, +)$; some authors use its max-plus dual semiring $(\mathbb{R}_{\max}, \vee, +)$. We use both semirings as part of the weighted lattice - clog $(\overline{\mathbb{R}}, \vee, \wedge, +)$. For example, the analytic expressions for the Euclidean line $y_{\text{e-line}} = ax + b$ and parabola $y_{\text{e-parab}} = ax^2 + bx + c$ become the tropical curves shown in Fig. 1 and described by the max-plus polynomials

$$y_{\text{t-line}} = \max(a + x, b), \quad y_{\text{t-parab}} = \max(a + 2x, b + x, c) \tag{18}$$

The above examples generalize to multiple dimensions or higher degrees and show us the way to tropicalize any classic n-variable polynomial (linear combination of power monomials) $\sum_i a_i z_1^{u_1^i} \cdots z_n^{u_n^i}$ defined over $\mathbb{R}^n$ where $\mathbf{u}^i = (u_1^i, ..., u_n^i)$ is some nonnegative integer vector: replace the sum with max and log the individual terms so that the multiplicative coefficients become additive and the powers become integer multiples of the indefinite log variables. Thus, a general max-polynomial $p : \mathbb{R}^n \to \mathbb{R}$ has the expression:

$$p(\mathbf{x}) = \bigvee_{i=1}^{k} b_i + \mathbf{c}_i^T \mathbf{x}, \quad \mathbf{x} = (x_1, ... x_n) \tag{19}$$



(a) Euclidean line (b) Tropical line (c) Euclid parabola (d) Tropic parabola

Fig. 1. Euclidean curves and their tropical versions.

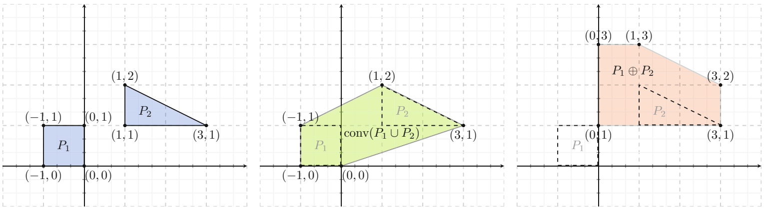
where k is the rank of p. Further, we can assume as in [6] real coefficient vectors $\mathbf{c}_i \in \mathbb{R}^n$. An interesting geometric object related to a max-polynomial p is its *Newton polytope* (conv$(\cdot)$ denotes convex hull)

$$\text{New}(p) \triangleq \text{conv}\{\mathbf{c}_i : i = 1, ..., \text{rank}(p)\} \tag{20}$$

This satisfies several important properties [7] (see Fig. 2 for an example):

$$\text{New}(p_1 \vee p_2) = \text{conv}(\text{New}(p_1) \cup \text{New}(p_2)) \tag{21}$$

$$\text{New}(p_1 + p_2) = \text{New}(p_1) \oplus \text{New}(p_2) \tag{22}$$



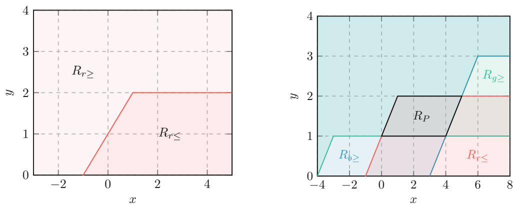
(a) Polytopes (b) Newton (max) (c) Newton (sum)

Fig. 2. (a) Newton polytopes of two max-polynomials $p_1(x, y) = \max(x+y, 3x+y, x+2y)$ and $p_2(x, y) = \max(0, -x, y, y-x)$, (b) their max $p_1 \vee p_2$, and (c) their sum p_1+p_2.

In pattern analysis problems on Euclidean spaces $\mathbb{R}^{n+1}$ we often use halfspaces $\mathcal{H}(\mathbf{a}, b) := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} \leq b\}$, polyhedra (finite intersections of halfspaces) and polytopes (compact polyhedra formed as the convex hull of a finite set of points). Replacing linear inner products $\mathbf{a}^T\mathbf{x}$ with max-plus versions yields *tropical halfspaces* [11] with parameters $\mathbf{a} = [a_i], \mathbf{b} = [b_i] \in \mathbb{R}^{n+1}$:

$$\mathcal{T}(\mathbf{a}, \mathbf{b}) \triangleq \{\mathbf{x} \in \mathbb{R}^n : \max(a_{n+1}, \bigvee_{i=1}^{n} a_i + x_i) \leq \max(b_{n+1}, \bigvee_{i=1}^{n} b_i + x_i)\} \tag{23}$$

where $\min(a_i, b_i) = -\infty \; \forall i$. Examples of regions formed by such tropical halfspaces are shown in Fig. 3. Obviously, their separating boundaries are tropical lines. Such regions were used in [7] as morphological perceptrons.

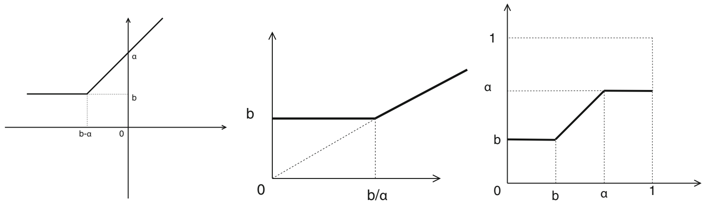


(a) Single region (b) Multiple regions

Fig. 3. Regions formed by tropical halfspaces in $\mathbb{R}^2$.

In the same way that weighted lattices generalize max-plus morphology and extend it to other types of clodum arithmetic, we can extend the above objects of max-plus tropical geometry to other max-$\star$ geometric objects. For example, over a clodum $(\mathcal{K}, \vee, \wedge, \star, \star')$, we can generalize tropical lines as $y = \max(a \star x, b)$ and tropical planes as $z = \max(a \star x, b \star y, c)$. Figure 4 shows some generalized tropical lines where the $\star$ operation is sum, product and min. Further, we can generalize max-plus halfspaces (23) to max-$\star$ tropical halfspaces:

$$\mathcal{T}(\mathbf{a}, \mathbf{b}) \triangleq \{\mathbf{x} \in \mathcal{K}^n : \mathbf{a}^T \boxed{\star} \begin{pmatrix} \mathbf{x} \\ e \end{pmatrix} \leq \mathbf{b}^T \boxed{\star} \begin{pmatrix} \mathbf{x} \\ e \end{pmatrix}\} \tag{24}$$



(a) max-plus line (b) max-product line (c) max-min line

Fig. 4. Max-$\star$ tropical lines $y = \max(a \star x, b)$: (a) $\star = +$, (b) $\star = \times$, (c) $\star = \wedge$.

6 Applications to Optimization and Machine Learning

6.1 Solving Max-$\star$ Equations

Consider a scalar clodum $(\mathcal{K}, \vee, \wedge, \star, \star')$, a matrix $\mathbf{A} \in \mathcal{K}^{m \times n}$ and a vector $\mathbf{b} \in \mathcal{K}^m$. The set of solutions of the max-$\star$ equation

$$\mathbf{A} \boxed{\star} \mathbf{x} = \mathbf{b} \tag{25}$$

over $\mathcal{K}$ is either empty or forms a sup-semillatice. A related problem in applications of max-plus algebra to scheduling is when a vector $\mathbf{x}$ represents start times, a vector $\mathbf{b}$ represents finish times and the matrix $\mathbf{A}$ represents processing delays. Then, if $\mathbf{A} \boxed{\star} \mathbf{x} = \mathbf{b}$ does not have an exact solution, it is possible to find the optimum $\mathbf{x}$ such that we minimize a norm of the earliness subject to zero lateness. We generalize this problem from max-plus to max-$\star$ algebra. The optimum will be the solution of the following constrained minimization problem:

$$\text{Minimize } \|\mathbf{A} \boxed{\star} \mathbf{x} - \mathbf{b}\| \;\text{ s.t. }\; \mathbf{A} \boxed{\star} \mathbf{x} \leq \mathbf{b} \tag{26}$$

where the norm $\|\cdot\|$ is either the ℓ_∞ or the ℓ_1 norm. While the two above problems have been solved in [10] for the max-plus case, we provide next a more general result using adjunctions for the general case when $\mathcal{K}$ is just a clodum or a general clog.

Theorem 3 *([21]).* *Consider a vector dilation $\delta(\mathbf{x}) = \mathbf{A} \boxast \mathbf{x}$ over a clodum $\mathcal{K}$ and let ε be its adjoint vector erosion. (a) If Eq. (25) has a solution, then*

$$\hat{\mathbf{x}} = \varepsilon(\mathbf{b}) = \mathbf{A}^T \square'_\zeta \mathbf{b} = [\bigwedge_i \zeta(a_{ij}, b_i)] \tag{27}$$

is its greatest solution, where ζ is the scalar adjoint erosion of $\star$ as in (10). (b) If $\mathcal{K}$ is a clog, the solution (27) becomes

$$\hat{\mathbf{x}} = \mathbf{A}^* \boxast' \mathbf{b} \tag{28}$$

(c) The solution to problem (26) is generally (27), or (28) in the case of a clog.

A main idea for solving (26) is to consider vectors $\mathbf{x}$ that are *subsolutions* in the sense that $\delta(\mathbf{x}) = A \boxast \mathbf{x} \leq \mathbf{b}$ and find the greatest such subsolution $\hat{\mathbf{x}} = \varepsilon(\mathbf{b})$, which yields either the greatest exact solution of (25) or an optimum approximate solution in the sense of (26). This creates a lattice projection onto the max-$\star$ span of the columns of $\mathbf{A}$ via the opening $\delta(\varepsilon(\mathbf{b})) \leq \mathbf{b}$ that best approximates $\mathbf{b}$ from below.

6.2 Regression for Optimal Fitting Tropical Lines to Data

We examine a classic problem in machine learning, fitting a line to data by minimizing an error norm, in the light of tropical geometry. Given data $(x_i, y_i) \in \mathbb{R}^2$, $i = 1, ..., n$, if we wish to fit a Euclidean line $y = ax + b$ by minimizing the ℓ_2 error norm, the optimal (least-squares) solution for the parameters a, b is

$$\hat{a}_{\text{LS}} = \frac{n \sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{n \sum_i (x_i)^2 - (\sum_i x_i)^2}, \quad \hat{b}_{\text{LS}} = \frac{1}{n} \sum_i (y_i - \hat{a}_{\text{LS}} x_i) \tag{29}$$

Suppose now we wish to fit a general tropical line $y = \max(a \star x, b)$ by minimizing the ℓ_1 error norm. The equations to solve become:

$$\underbrace{\begin{bmatrix} x_1 & e \\ \vdots & \vdots \\ x_n & e \end{bmatrix}}_{\mathbf{X}} \boxast \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}}_{\mathbf{y}} \tag{30}$$

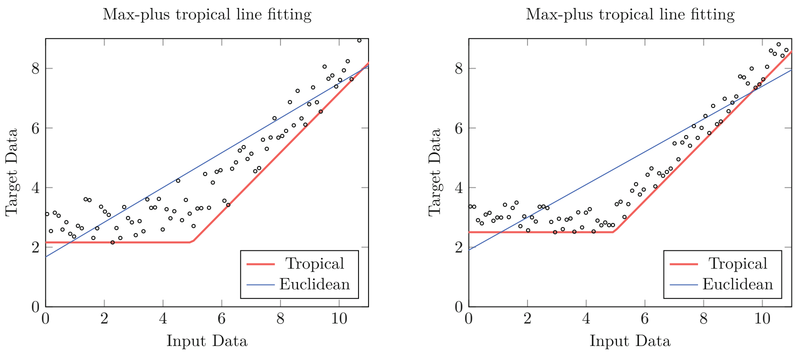
By Theorem 3, the optimal (min ℓ_1 error) solution for any clodum arithmetic is

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \mathbf{X}^T \square'_\zeta \mathbf{y} = \begin{bmatrix} \bigwedge_i \zeta(x_i, y_i) \\ \bigwedge_i \zeta(e, y_i) \end{bmatrix} \tag{31}$$

where ζ is the scalar adjoint erosion (10) of $\star$. If $\mathcal{K}$ is a clog like in the max-plus and max-product case, then $\zeta(a, w) = a^* \star' w$. Next we write in detail the solution for the tropical line for the three special cases where the scalar arithmetic is based on the max-plus clog, max-product clog and the max-min clodum (the shapes of the corresponding lines are shown in Fig. 4):

$$(\hat{a},\ \hat{b}) = \begin{cases} \bigwedge_i y_i - x_i,\ \bigwedge_i y_i), & \text{max-plus } (\star = +) \\ \bigwedge_i y_i / x_i,\ \bigwedge_i y_i), & \text{max-times } (\star = \times) \\ \bigwedge_i \max([y_i \geq x_i], y_i),\ \bigwedge_i y_i), & \text{max-min } (\star = \wedge) \end{cases} \tag{32}$$

where $[\cdot]$ denotes Iverson's bracket in the max-min case. Thus, the above approach allows to optimally fit tropical lines to arbitrary data. Figure 5 shows an example. It can also be generalized to higher-degree curves and to high-dimensional data.



(a) T-line with Gaussian Noise (b) T-line with Uniform Noise

Fig. 5. (a) Red curve: Optimal fitting via (32) of a max-plus tropical line to data $y = \max(x - 2, 3)$ corrupted by additive i.i.d. Gaussian noise $N(0, 0.25)$. Blue line: Euclidean line fitting via least squares. (b) Same experiment as in (a) but with uniform noise $U(-0.5, 0.5)$. (Color figure online)

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