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A UNIFICATION OF LINEAR, MEDIAN, ORDER-STATISTICS AND MORPHOLOGICAL FILTERS UNDER MATHEMATICAL MORPHOLOGY*

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ABSTRACT - *This paper presents a summary of a unified theory, based upon mathematical morphology, of all translation-invariant and increasing systems. Examples of such systems are morphological transformations of signals, order-statistics filters and some linear shift-invariant filters. Our theoretical research showed that every such system can be uniquely represented by the minimal elements of its kernel and realized as a minimal combination of morphological erosions.*

INTRODUCTION

Mathematical Morphology was introduced by Matheron and Serra [1,2] as a set-theoretical method for image analysis whose purpose is the *quantitative description of geometrical structures*. Using mathematical morphology as a tool, our theoretical research aims at studying the geometrical structure of signals and systems. A n-dimensional signal can be mathematically represented by a function of n independent variables. The function representing a n-dimensional signal may assume only binary values, e.g. binary images, in which case we can represent the signal as a mathematical set in a n-dimensional Euclidean space. Henceforth, functions and sets will be viewed as special cases of mathematical representations for signals with the distinction that, function implies a multi-valued signal whereas set refers to a binary-valued signal. We adopt a similar classification for systems by considering function-processing and set-processing systems. A n-dimensional system is called set-processing if it can accept n-dimensional binary-valued signals as inputs and produce n-dimensional binary-valued signals as outputs, e.g. a system processing binary images. Mathematical morphology represents image objects as sets in a Euclidean space. Thus, in our analysis, the set is the primary notion and function is a particular case; e.g. a n-dimensional multi-valued function is viewed as a set in a (n+1)-dimensional space. In this light then, any function- or set-processing system is viewed as a set mapping (transformation) from one class of sets into another class of sets. However, the concept of a set is more general than needed to represent an image object. A good compromise (Matheron[1]) is to select the class of all closed subsets of a Euclidean space E, denoted by F(E), to represent image objects. The generalized space F(E), topologized by the *Hit-or-Miss*[1,2] topology, becomes a compact, Hausdorf, topological space with a countable base, which enables us to study convergence and continuity inside it.

Mathematical morphology extracts information from image objects by first choosing a structuring element, which is another object of rather simpler shape and size than the original object. The structuring element interacts with the image object and transforms it into another more expressive form. These morphological transformations of image objects by structuring elements, in order to be quantitative, must satisfy four principles[2], two of which are invariance under vector translation and upper-semicontinuity of the set mappings inside the class F(E). Before we present our new results, we briefly define and comment on morphological transformations of sets and functions, order-statistics filters and the kernel representation of translation-invariant systems. But first we introduce some notation:

NOTATION

R = set of real numbers; Z = set of integers;
E = Euclidean spaces R^n, Z^n, $Z^n \times R$,...;
D = R^n or Z^n: domain of definition of functions f,g
F(E) = class of all closed subsets of E;
X,A,B=subsets of E or D; f,g=functions defined on D
X^c = complement of X; $|X|$ = cardinality of X;
{x: P} = set of points x satisfying a property P
$\cup(\cap)$ = set intersection (union)
$A \subseteq B$ = set A is a subset of set B
$[f \wedge g](x) = INF[f(x),g(x)]$, $x \in D$
$[f \vee g](x) = SUP[f(x),g(x)]$, $x \in D$
$f \le g = f(x) \le g(x)$, all $x \in D$
$B_z = \{z+b: b \in B\}$ = translate of B by the vector $z \in E$

SETS

EROSION: $X \ominus B = \{z: B_z \subseteq X\} = \bigcap_{b \in B} X_{-b}$ (1)

DILATION: $X \oplus B = \{z: B_z \cap X \neq \emptyset\} = \bigcup_{b \in B} X_{-b}$ (2)

OPENING: $X_B = (X \ominus B) \oplus B$; CLOSING: $X^B = (X \oplus B) \ominus B$ (3)

CROSS-SECTION: $X_t(f) = \{x \in D: f(x) \ge t\}$ (4)

UMBRA: $U(f) = \{(x,t) \in D \times R: f(x) \ge t\}$ (5)

K-TH ORDER-STATISTIC: $(X : B)^k = \{z: |X \cap B_z| \ge k\}$ (6)

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FUNCTIONS

$$[f\ominus B](x) = \mathrm{INF}\{f(z):\ z\in B_x\}, \qquad \text{all } x\in D \qquad (7)$$

$$[f\oplus B](x) = \mathrm{SUP}\{f(z):\ z\in B_x\}, \qquad \text{all } x\in D \qquad (8)$$

$$[f\ominus g](x) = \mathrm{INF}\{f(z)-g(x-z):\ z\in D\}, \qquad \text{all } x\in D \qquad (9)$$

BACKGROUND

A. Morphological Transformations: Let the closed set X represent an image object and the compact set B a structuring element. The basic morphological transformations of X by B are the erosion, dilation, opening and closing. The erosion of X by B (see Eq.1 and Fig.1) is the set of all points z such that the translate B_z is included in X. The dilation of X by B (see Eq.2) is the set of all points z such that the translate B_z of B intersects X. Figure 1 shows that erosion shrinks the object whereas dilation expands it. Erosion and dilation are dual operations with respect to complementation, meaning that dilating an image object is equivalent to eroding its background. Another pair of dual morphological transformations is the opening and the closing: If we dilate the erosion $X\ominus B$ by B, we do not recover X generally; we obtain the opening X_B of X by B. By duality, the closing of X by B results from dilating first and then eroding (see Eqs.3). As Fig.1 shows, both opening and closing are nonlinear filters which smooth the contours of X in a way such that always $X_B\subseteq X\subseteq X^B$.

The above set transformations are generalized to functions by establishing first a link between sets and functions using two different concepts: the cross-sections and the umbra of a function (see Fig.2). The cross-section $X_t(f)$ of a function f(x), $x\in D$, at level t is a subset of D consisting of those points x such that $f(x)\geq t$ (see Eq.4). The cross-sections of a function form a family of decreasing sets. The umbra U(f) of the function f (see Eq.5) is a subset of $E=D\times R$, and it occupies all the space in E which extends below the graph of f down to $-\infty$. Recall that our working space is the class of all closed subsets of E. Hence, the equivalent class of functions, whose cross-sections are closed sets in D, or, equivalently, whose umbras are closed sets in E, is the class of uppersemicontinuous functions on D, abbreviated as u.s.c. The mapping between u.s.c. functions and their umbras is a topological mapping; i.e. it is one-to-one and onto and continuous in both directions. As a result, the set intersection and union, by which erosion and dilation of sets are also defined in Eqs.(1),(2), correspond to the infimum "$\wedge$" and supremum "$\vee$" respectively between functions. Also, the set inclusion ($A\subseteq B$) corresponds to the ordering of functions ($f\leq g$). Then, the erosion and dilation of a function f by a set B are defined in Eqs.(7),(8) as the infimum and supremum respectively of the function f(x) inside the translated set B_x. Erosion and dilation of functions by sets commute with threshholding (taking cross-sections). For example,

$$X_t(f\ominus B) = [X_t(f)]\ominus B \qquad (10)$$

Similar results are valid for the opening f_B and closing of f by B. Thus, morphological transformations of a function by a set are function-processing systems which can be analyzed and realized as set-processing systems. The erosion of a function enlarges its minima, the dilation enlarges its maxima, the opening cuts down its peaks, and the closing fills up its valleys (see Fig.3). Finally, the last step of generalization is to morphologically transform a function by another structuring function: The erosion of a function f by another function g is defined in Eq.(9) using the infimum and an additive convolution between f and g. Using supremum and "+" in Eq.(9), instead of infimum and "-", gives us the dilation of f by g.

For all the above morphological operations we assumed that both the structuring set B and the structuring function g are symmetric with respect to the origin; otherwise, the definitions of the operations are slightly more complicated. In addition, B must be a compact set and g must have a compact region of support so that all the above morphological transformations are upper-semicontinuous mappings. This last simplification allows the erosion and dilation of discrete sets to be realized by finite intersections and unions respectively, and the erosion and dilation of sampled functions as local MIN and MAX operations. The openings and closings of functions have been called *"M-FILTERS"*[3]. We call all the morphological transformations of sets and functions *morphological systems or filters*.

B. Order-Statistics: Order-statistics filters for functions have been defined as a generalization of median filters[4,5]. Let f(x), $x\in Z^n$, be a n-dimensional sampled function and B is a bounded subset of Z^n with $|B|=N$. Then, the **k-th order-statistic** of f with respect to B, denoted by $(f:B)^k$, is another function whose value at x is

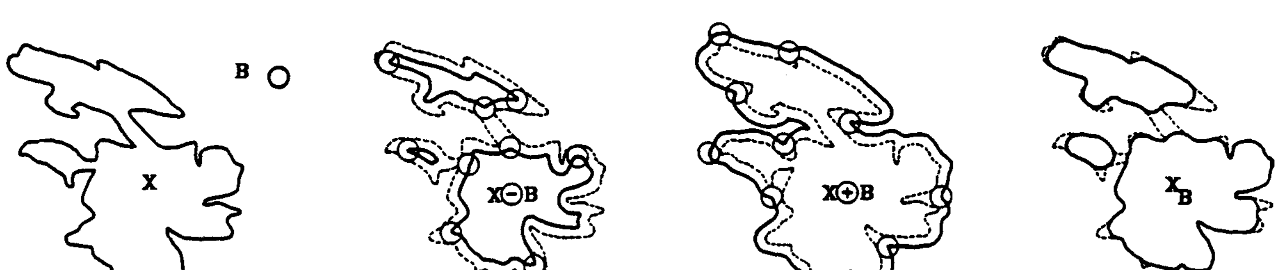

Figure 1 - Erosion, dilation, opening and closing of X by B (the dark solid curve refers to the transformed object and the dashed curve to the original object).

obtained by sorting in descending order the N values of f inside the window B shifted to location x and picking the k-th number from the sorted list, k=1,2,...,N. For k=(N+1)/2 and N odd, we have the case of the median of a function. In Eq.(6) we define the k-th order-statistic of a set X with respect to B, which is obtained by counting points instead of sorting. These definitions allow us to identify the first and N-th order-statistics of both functions and sets with their erosion and dilation respectively by the set B. Moreover, the order-statistics filters for functions commute with threshholding; i.e.

$$X_t[(f : B)^k] = [X_t(f) : B]^k \tag{11}$$

C. Kernels of τ-systems: Consider the set-processing system (mapping) T: X→T(X). It is called translation-invariant, or a **τ-system** by Matheron, if it commutes with vector translation; i.e. $T(X_z)=[T(X)]_z$. Such systems are uniquely represented and realized by their **kernel** K(T):

$$K(T) = \{ X : T(X) \text{ contains the origin } 0 \} \tag{12}$$

The kernel is a collection of sets. If the τ-system T is **increasing**, i.e. if $A \subseteq B$ implies $T(A) \subseteq (B)$, then the kernel has a special structure, and the following theorem results:

Theorem 1 (Matheron): *All set-processing increasing τ-systems can be realized as a union of erosions by all the elements of its kernel.*

NEW RESULTS

We extended the kernel representation to function-processing systems: A vector translation of the umbra of a function corresponds uniquely to a shift of both the argument and the amplitude of the function. Thus, function-processing τ-systems are those which commute with vector translation of functions. Linear shift-invariant filters with dc-gain=1 are τ-systems. The kernel K(T) of a function-processing τ-system T: f→T(f) is the following collection of functions:

$$K(T) = \{ f : [T(f)](0) \geq 0 \} \tag{13}$$

The function-processing system T is increasing if $f \leq g$ implies $T(f) \leq T(g)$. Linear shift-invariant systems possessing a nonnegative impulse response are increasing.

Theorem 2: *Any function-processing increasing τ-system can be realized as a supremum of erosions by all the functions of its kernel.*

A similar result, as above, is mentioned without proof in [3].

Theorems 1,2 are of no practical use because they require the spanning of all the kernel elements, which are infinite in number. However, if we can find some **minimal** kernel elements, then we can significantly reduce the number of erosions required to realize the system. The kernel of a set- or function-processing τ-system is partially ordered with respect to set inclusion "⊆" or ordering of functions "≤", and a kernel element is **minimal** if it is not preceded by any other kernel element. Thus, we define the **basis** to be the set of minimal elements of the kernel. By its definition, the basis is a subcollection of the kernel, and it may sometimes be finite. The existence of such a basis is proved by the following two theorems[6]. Let A be a subclass of F(E) closed under translation and infinite intersection. Then,

Theorem 3: *If T: A→F(E) is an increasing u.s.c. set-processing τ-system, then, its kernel has a minimal element; i.e. its basis is nonempty.*

Theorem 4: *The basis of any increasing u.s.c. τ-system processing u.s.c. functions is nonempty.*

Now, for the realization of such systems we proved the following two theorems[6]:

Theorem 5: *Any increasing u.s.c. set- or function-processing τ-system is the union or supremum respectively of erosions by all the elements of its basis.*

Theorem 6: *Let T be an increasing u.s.c. set-processing τ-system and T^* is its* dual *system with respect to complementation; i.e. $T^*(X)=[T(X^c)]^c$. Then the system T can be realized not only as union of erosions by its own basis elements (Theorem 5), but also as an intersection of dilations by the basis elements of its dual system T^*.*

We also have found[6,7] some interesting relations between morphological and order-statistics filters: 1) Any order-statistics filter for sets (resp. functions) can be realized as a finite union (resp. maximum) of erosions or as a finite intersection (resp. minimum) of dilations. 2) Medians of sets and functions are bounded between morphological openings and closings. 3) A 1-D signal is a median root[4] with respect to a convex window of 2N+1 points if and only if it is both a root of an opening and closing by a convex window of N+1 points. Similar results have been found for 2-D sampled signals. 4) An opening followed by a

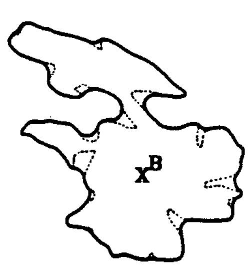

Figure 1 (continued)

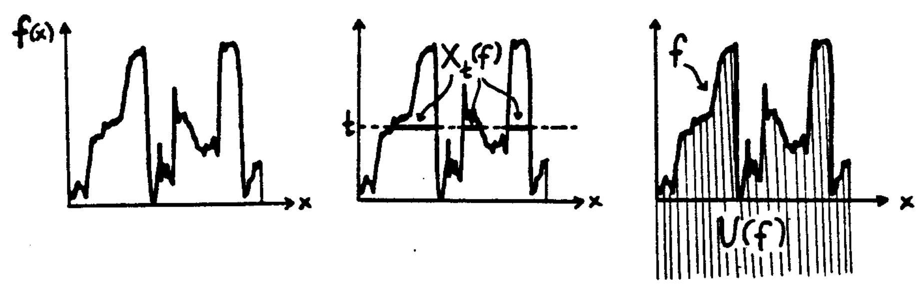

Figure 2 - A function f, its cross-section at level t and its umbra.

closing, or vice-versa, gives a median root in only one iteration.

Individual parts of all the above theoretical research with proofs are being prepared for more detailed publications[6,7].

Example 1: *3-POINT 1-D MEDIAN FILTER*
Referring to Eq.(6), $B=\{-1,0,1\}$ and $|B|=3$. Its kernel, as a set-processing system, is the collection of all subsets X of Z such that $|X\cap B|\geq 2$. The kernel has 3 minimal elements which are subsets of B: $M_1=\{-1,0\}$, $M_2=\{0,1\}$ and $M_3=\{-1,1\}$. Erosion of a function f(n), $n\in Z$, by M_1 is the minimum of f inside the window M_1 shifted to location n, and similarly for M_2 and M_3. Hence, see Theor.5, the median of f with respect to B is:

$$\text{med}(f : B) = \text{MAX}\left\{\begin{array}{c}\text{MIN}[f(n-1),f(n)]\\ \text{MIN}[f(n),f(n+1)]\\ \text{MIN}[f(n-1),f(n+1)]\end{array}\right\} \quad (14)$$

Because the median commutes with set complementation, we can interchange MIN and MAX in (14).

Example 2: *1-D FIR DISCRETE LINEAR FILTER*
Let the impulse response h(n) of the linear filter have the value of 0.5 at n=0,1 and zero everywhere else. Then this filter is an increasing u.s.c. τ-system. Its kernel consists of all the functions f such that $h*f(0)\geq 0$, where "*" denotes discrete convolution. The minimal elements are the functions g such that $g(0)=\alpha$, $g(-1)=-\alpha$, $\alpha\in R$, and $g(n)=0$ or $-\infty$ for convolution or SUP-operation respectively wherever $h(-n)=0$. Then, see Theor.5, if f(n) is an input function, the output $h*f(n)$ is equal to:

$$0.5[f(n)+f(n-1)] = \text{SUP}\{\text{MIN}[f(n)-\alpha, f(n-1)+\alpha] : \alpha\in R\}$$

Thus, a linear convolution was expressed in terms of minimum and supremum.

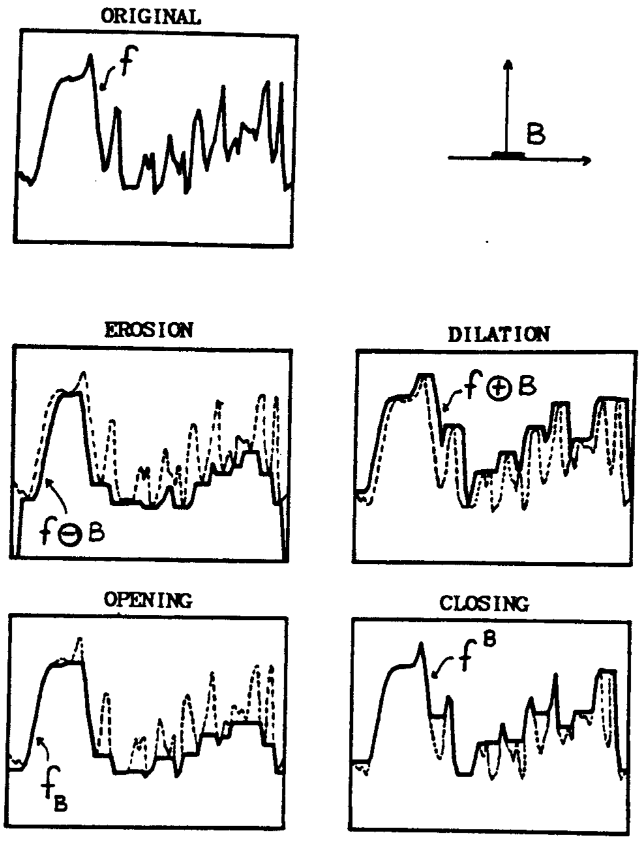

Figure 3 - Erosion, dilation, opening and closing of a function f by a set B (the dashed curve refers to the original function).

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