THRESHOLD PARALLELISM IN MORPHOLOGICAL FEATURE EXTRACTION, SKELETONIZATION, AND PATTERN SPECTRUM

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Abstract

In this paper it is shown that many composite morphological systems, such as morphological edge detection, peak/valley extraction, skeletonization, and shape-size distributions obey a weak linear superposition, called *threshold-linear superposition:* Namely, the output graytone image is the sum of outputs due to input binary images, which result from thresholding the input graytone image at all levels. Then these results are generalized to a vector space formulation, e.g., to any linear combination of simple morphological systems such as erosion, dilation, rank-order filters, and their cascade or max/min combinations. Thus many such systems processing graytone images are reduced to corresponding binary image processing systems, which are easier to analyze and implement.

1 Introduction

Morphological image analysis systems [1]-[15] are useful for feature extraction, shape analysis, and nonlinear filtering. A major limitation, though, in their theoretical analysis and application has been so far the nonlinearity of the signal operations involved. Specifically, the morphological image operations do not obey the well-known *additive superposition* principle, which is obeyed by all linear systems. However, a special class of morphological operations, in particular the erosions, dilations, openings, closings that can process both graytone and binary images without changing this signal characteristic, obey a weak additive superposition: Namely, if the input graytone image is expressed as the sum of all its binary threshold versions, then the output image from any of these filters is the sum of the filtered input threshold binary images. We call this system property *threshold-linear superposition.* Such ideas have been proven very useful in analyzing and implementing morphological filters [2,4,10] and median-type filters [11],[16]-[20].

In practice, the useful morphological image analysis systems do not consist of individual erosions, dilations, openings, and closings, but they include parallel and/or series interconnections of simple morphological operations. For example, 1) the morphological peak/valley extractor involves an (algebraic) difference between the image and its opening [4]. 2) The morphological edge detection involves the difference between the image and its erosion [5,12,21,22]. 3) The graytone skeleton is the sum of components, each of which is the difference between erosions and openings [2,8,9]. 4) The graytone pattern spectrum involves areas of differences among openings or closings by structuring elements of varying size [13].

In this paper, we show that all the four above composite morphological systems obey the threshold-linear superposition. That is, given any input graytone image, their outputs are the sum of the individual system outputs corresponding to input binary images that resulted from exhaustive thresholding of the input image. The processing of these threshold binary images is much easier to analyze and implement. Thus our results offer new tools for the theoretical analysis of these nonlinear systems and suggest new parallel implementations since the processing of the threshold binary images can take place in parallel at all threshold levels simultaneously. Finally, we generalize the above results by showing that the four above systems together with any other system that obeys threshold-linear superposition form a vector space.

2 Preliminaries

Consider a digital graytone image signal represented by a nonnegative 2-D sequence $f(m,n)$, which assumes $A+1$ possible intensity values: $a = 0, 1, 2, \ldots, A$. For example, if we deal with 8 bit/pixel imagery, $A = 255$. By thresholding f at all possible amplitude levels $0 \leq a \leq A$ we obtain the ***threshold binary images***

$$f_a(m,n) = \begin{cases} 1 & , \quad f(m,n) \geq a \\ 0 & , \quad f(m,n) < a \end{cases} \tag{1}$$

If there is a risk of notational confusion, we will also denote the signal f_a by $t_a(f)$. It is simple to show that f can be reconstructed exactly from all its binary thresholded versions; i.e. $\forall\,(m,n)$

$$f(m,n) = \max\{a : f_a(m,n) = 1\} = \sum_{a=1}^{A} f_a(m,n) \tag{2}$$

In this paper, by a ***system*** Ψ processing an input image f we mean either an ***image transformation*** where the system output $\Psi(f)$ is an image signal, or an ***image measurement***. In the latter case $\Psi(f)$ is either a real number (e.g., the area of the image) or a real function of several parameters measuring some characteristics of the image. We shall say that Ψ ***commutes with thresholding*** if Ψ is an image transformation such that

$$\Psi[t_a(f)] = t_a[\Psi(f)] \tag{3}$$

for any input image f and any amplitude level a. Note that a necessary condition for Ψ to obey (3) is, whenever Ψ processes a binary image, to leave this signal characteristic unchanged. Thus if a system Ψ commutes with thresholding, processing by Ψ the threshold binary image f_a gives the same result with processing first by Ψ the graytone image f and then thresholding $\Psi(f)$ at level a. For example, the basic morphological transformations of ***erosion*** $f \ominus B$ of an image f by a 2-D structuring set (finite window) B, ***dilation*** $f \oplus B$, ***opening*** $f \circ B$, and ***closing*** $f \bullet B$, which are defined[1] below, commute with thresholding [2,3].

$$(f \ominus B)(m,n) = \min\{f(m+i, n+j) : (i,j) \in B\} \tag{4}$$

$$(f \oplus B)(m,n) = \max\{f(m-i, n-j) : (i,j) \in B\} \tag{5}$$

$$f \circ B = (f \ominus B) \oplus B \tag{6}$$

$$f \bullet B = (f \oplus B) \ominus B \tag{7}$$

Thus, $t_a(f \ominus B) = f_a \ominus B$, where the notation $x = y$ for two signals means $x(m,n) = y(m,n)\ \forall\,(m,n)$.

We shall say that a system Ψ obeys the ***threshold-linear superposition*** provided that

$$\Psi(f) = \sum_{a=1}^{A} \Psi(f_a) \tag{8}$$

for any input image f. (Although f_a is binary, note that $\Psi(f_a)$, if it is an image signal, could be binary or multilevel.) Such a system Ψ can be realized by decomposing f into all its threshold binary images f_a, processing them by Ψ, and create the output $\Psi(f)$ by adding the processed f_a. A fundamental motivation for such a realization of Ψ is that, due to their binary range, the processing of the f_a's by Ψ is easier to analyze and implement than the processing of f.

[1] In the recent literature on morphology, there are mainly two slightly different sets of definitions for morphological operations: one of [1,2] and another of [7,14], which become identical if B is symmetric. Maragos and Schafer used in [9]-[13] the definitions from Matheron & Serra. In this paper we use Sternberg's definitions and the notation of Haralick *et al.* because they are simpler.

The morphological image transformations (4)-(7) of erosion, dilation, opening, and closing obey a *threshold max-superposition* [2]:

$$[\Psi(f)](m,n) = \max\{a : [\Psi(f_a)](m,n) = 1\} \tag{9}$$

This max-superposition is also obeyed by median and rank-order filters [16,11]. However, these simple morphological and median-type systems obey both the threshold sum-superposition (8) and the max-superposition (9) because they commute with thresholding. Thus a sufficient (but not necessary) condition for threshold superposition is commuting with thresholding. From one viewpoint, the threshold max-superposition is more general than the sum-superposition since the latter applies only to nonnegative input signals, while the former applies to any real-valued input signals. From a different viewpoint, the max-superposition restricts the class of systems since it requires that $\Psi(f_a)$ are signals and binary, an assumption not needed by the sum-superposition. In addition, the threshold sum-superposition ties well with linear systems, because it is just a weak form of linear superposition. This last viewpoint will be instrumental for our analysis throughout the rest of this paper. Therefore, we focus henceforth on systems obeying (8).

3 Special Cases

3.1 Morphological Edge Detection

Given a graytone image $f(m,n)$ and a small 2-D symmetric structuring set K containing the origin, the simple system [5,12]

$$ED(f) = f - (f \ominus K) \tag{10}$$

produces a graytone image $ED(f)$ with enhanced edges, where $-$ denotes pointwise subtraction. A binary edge map can be obtained by thresholding $ED(f)$, which is nonnegative everywhere because K contains the origin. This simple but effective morphological edge detection system has been made more robust in [21,22] by incorporating some smoothing filters.

Now, because the erosion $f \ominus K$ satisfies (8), using the threshold decomposition (1) of f yields

$$\begin{aligned} ED(f) &= ED\left(\sum_{a=1}^{A} f_a\right) \\ &= \left(\sum_{a=1}^{A} f_a\right) - \left[\left(\sum_{a=1}^{A} f_a\right) \ominus K\right] \\ &= \sum_{a=1}^{A} f_a - \sum_{a=1}^{A} f_a \ominus K = \sum_{a=1}^{A} [f_a - (f_a \ominus K)] \\ &= \sum_{a=1}^{A} ED(f_a) \end{aligned} \tag{11}$$

Thus the morphological edge detection system (10) obeys the threshold-linear superposition. An example is given in Fig. 1. Note that, since $f_a \ominus K$ and $f_a - (f_a \ominus K)$ are binary images for all a, the binary edge detections $ED(f_a)$ can be implemented very simply by using only *pixel counting*. Namely, if $|\cdot|$ denotes set cardinality (i.e., number of pixels), then

$$[ED(f_a)](m,n) = \begin{cases} 1 \ , & \text{if } f_a(m,n) = 1 \text{ and } |\{(i,j) : f_a(i,j) = 1,\ (i-m, j-n) \in K\}| < |K| \\ 0 \ , & \text{otherwise} \end{cases}$$

Another edge detection system similar to (10) is $ED(f) = (f \oplus K) - (f \ominus K)$, which also obeys (8).

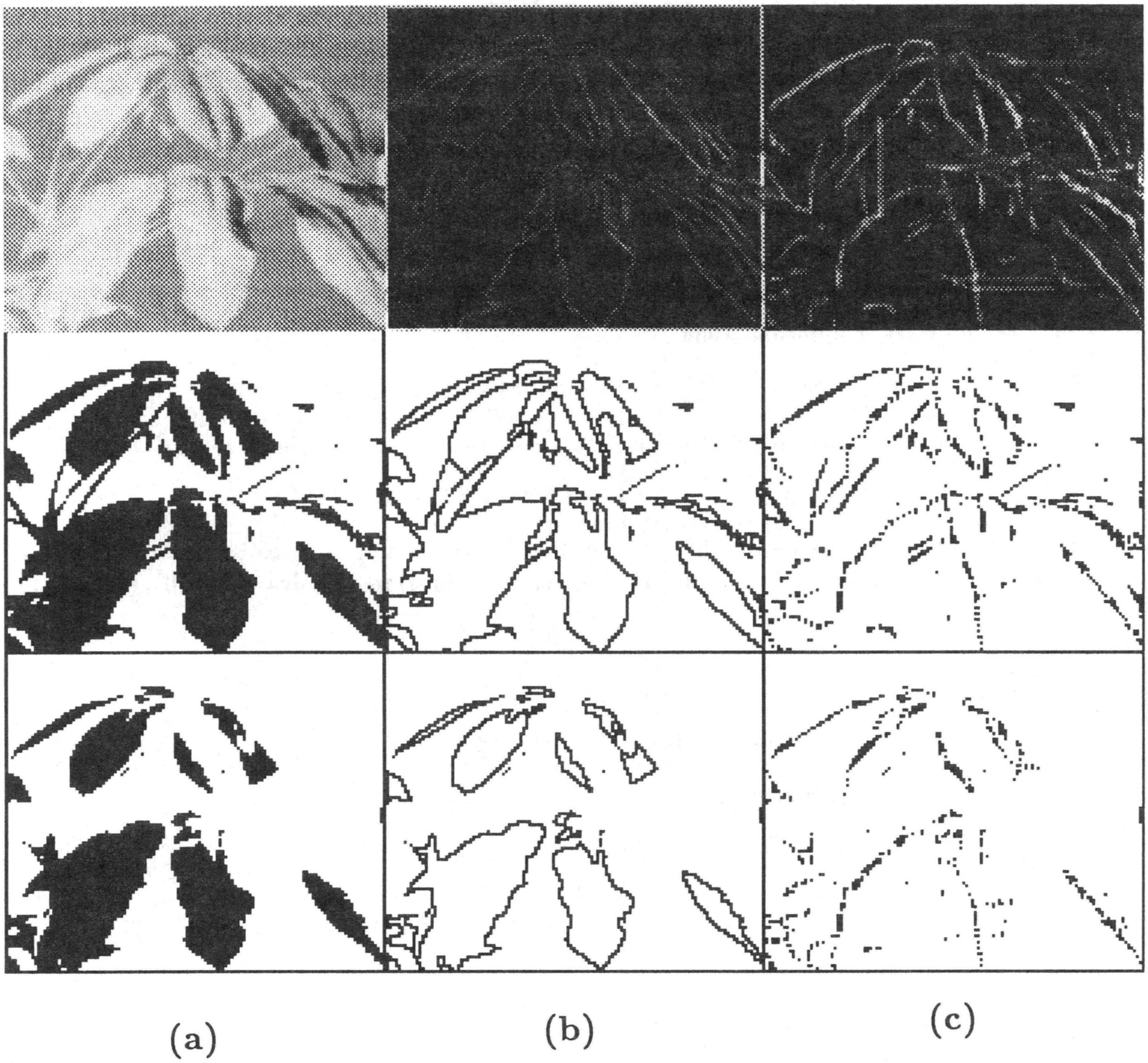

Figure 1. The top row (from left to right) shows an original graytone image f of 110×128 pixels with 8 bit/pixel, the graytone edge image $f - (f \ominus K)$, and the graytone skeleton $SK(f)$ with respect to K, where K is a 3×3-pixel structuring set. The other images show (from middle to bottom row): (a) threshold binary images f_a for $a = 180$ and 210. (b) their binary edge images $f_a - (f_a \ominus K)$. (c) their binary skeletons $SK(f_a)$. (In the top row the edge and skeleton image amplitude has been magnified; in the middle and bottom rows, the black (white) areas correspond to image foreground (background).)

3.2 Peak Extraction

Meyer [4] introduced a very useful morphological peak extractor, also called ***top-hat transformation***:

$$PE(f) = f - (f \circ B), \tag{12}$$

where B is any 2-D structuring set (the "base of the hat"). During this peak extraction, only those peaks whose base contains B remain; the rest get eliminated. For any B, $f \geq f \circ B$ everywhere; hence, $PE(f)$ is a nonnegative image signal. Since the opening $f \circ B$ obeys (8),

$$\begin{aligned} PE(f) &= \left(\sum_{a=1}^{A} f_a\right) - \left[\left(\sum_{a=1}^{A} f_a\right) \circ B\right] \\ &= \sum_{a=1}^{A} f_a - \sum_{a=1}^{A} f_a \circ B = \sum_{a=1}^{A} [f_a - (f_a \circ B)] \\ &= \sum_{a=1}^{A} PE(f_a) \end{aligned} \tag{13}$$

As an example, consider the 1-D image $f(m)$

$$f = \ldots 0\ 2\ 1\ 2\ 3\ 4\ 0\ 4\ 4\ 1\ 2\ 3\ 2\ 1\ 0 \ldots$$

where ... denotes infinite sequence of trailing zero values. If we want to extract from f all peaks with a width less than 3 pixels, we select $B = \{0, 1, 2\}$. Then the graytone opening is

$$f \circ B = \ldots 0\ 1\ 1\ 2\ 2\ 2\ 0\ 1\ 1\ 1\ 2\ 2\ 2\ 1\ 0 \ldots$$

and the graytone peak extraction gives the peak image

$$PE(f) = \ldots 0\ 1\ 0\ 0\ 1\ 2\ 0\ 3\ 3\ 0\ 0\ 1\ 0\ 0\ 0 \ldots$$

Now the threshold binary images of f are f_a, $1 \leq a \leq 4$:

$$\begin{aligned} f_4 &= \ldots 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0 \ldots \\ f_3 &= \ldots 0\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0 \ldots \\ f_2 &= \ldots 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0 \ldots \\ f_1 &= \ldots 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0 \ldots \end{aligned}$$

and $f_0(m) = 1$ for all m. The binary openings $f_a \circ B$ are

$$\begin{aligned} f_4 \circ B &= \ldots 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \ldots \\ f_3 \circ B &= \ldots 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \ldots \\ f_2 \circ B &= \ldots 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0 \ldots \\ f_1 \circ B &= \ldots 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0 \ldots \end{aligned}$$

The binary peak extractions $PE(f_a) = f_a - (f_a \circ B)$ are

$$\begin{aligned} PE(f_4) &= \ldots 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0 \ldots \\ PE(f_3) &= \ldots 0\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0 \ldots \\ PE(f_2) &= \ldots 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0 \ldots \\ PE(f_1) &= \ldots 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \ldots \end{aligned}$$

Thus summing the signals $PE(f_a)$ for all a gives us the original signal $PE(f)$. Clearly, the binary peak extractors are trivial to implement. $PE(f_a)$ simply consists of eliminating from the binary image f_a all connected components that contain any shifted version of B. The implementation involves binary erosion/dilation and binary subtractions; hence, only pixel counting.

If we also consider the *valley extractor* system $VE(f) = (f \bullet B) - f$, by working as above, it can be shown that $VE(f) = \sum_a VE(f_a)$.

3.3 Skeletonization

A morphological skeleton for a graytone image f can be defined [2,8,9] as follows. If B is a 2-D structuring set, let $nB = B \oplus B \oplus \cdots \oplus B$ denote the n-fold dilation of B with itself, which creates a set of size $n = 0, 1, 2, \ldots$ times larger than B. The n^{th} ***skeleton component*** of f is

$$SK_n(f) = (f \ominus nB) - [(f \ominus nB) \circ B] \quad , \quad 0 \leq n \leq N \tag{14}$$

where $N = \max\{n : f \ominus nB \neq 0\}$. (We assume here images f with a finite support.) These components $SK_n(f)$ indexed by the discrete size parameter n, are nonnegative everywhere. Thus they are graytone images, usually very sparse, and their ensemble can exactly reconstruct f. A skeleton, i.e., a thinned caricature, of f can be defined as the graytone image

$$SK(f) = \sum_{n=0}^{N} SK_n(f) \tag{15}$$

Since erosions and openings of the binary images f_a by sets B of dimensionality ≤ 2 yields binary outputs and since $f_a \ominus nB \geq (f_a \ominus nB) \circ B$, the skeleton component, $SK_n(f_a)$, of f_a is also a binary image. The skeleton, $SK(f_a)$, of f_a is defined [2,9] as the union of all the binary skeleton components $SK_n(f_a)$, represented by 2-D sets. But this union- definition of $SK(f_a)$ is equivalent to a sum- definition as in (15) because the binary images $SK_n(f_a)$ are disjoint [9]. Putting all these ideas together yields

$$\begin{aligned} SK_n(f) &= \left[\left(\sum_{a=1}^{A} f_a\right) \ominus nB\right] - \left[\left(\sum_{a=1}^{A} f_a\right) \ominus nB\right] \circ B \\ &= \left(\sum_{a=1}^{A} f_a \ominus nB\right) - \left[\sum_{a=1}^{A} (f_a \ominus nB) \circ B\right] \\ &= \sum_{a=1}^{A} [(f_a \ominus nB) - (f_a \ominus nB) \circ B] \\ &= \sum_{a=1}^{A} SK_n(f_a) \end{aligned} \tag{16}$$

Thus, the n^{th} skeleton component system obeys the threshold superposition. Now,

$$\begin{aligned} SK(f) &= \sum_{n=o}^{N} SK_n\left(\sum_{a=1}^{A} f_a\right) = \sum_{n=0}^{N}\sum_{a=1}^{A} SK_n(f_a) = \sum_{a=1}^{A}\sum_{n=0}^{N} SK_n(f_a) \\ &= \sum_{a=1}^{A} SK(f_a) \end{aligned} \tag{17}$$

Hence, the morphological skeleton system also obeys threshold superposition. An example is given in Fig. 1.

3.4 Pattern Spectrum

The pattern spectrum of a graytone image $f(i,j)$ is defined in [13] as the nonnegative function

$$\begin{aligned} [PS(f)](+n,B) &= \sum_i \sum_j [f \circ nB - f \circ (n+1)B](i,j)\,, \quad n \geq 0 \\ [PS(f)](-n,B) &= \sum_i \sum_j [f \bullet nB - f \bullet (n-1)B](i,j)\,, \quad n > 0 \end{aligned} \tag{18}$$

where the integer n is a discrete size parameter and B is any 2-D structuring set whose shape can vary. Thus the pattern spectrum measures the size (n) and shape (B) distribution of f, giving us useful information about critical scales and the general shape-size content of f. Hence, for $n \geq 0$,

$$\begin{aligned} [PS(f)](n,B) &= \sum_i \sum_j (f \circ nB)(i,j) - \sum_i \sum_j [f \circ (n+1)B](i,j) \\ &= \sum_i \sum_j \left[\left(\sum_{a=1}^{A} f_a\right) \circ nB\right](i,j) - \sum_i \sum_j \left[\left(\sum_{a=1}^{A} f_a\right) \circ (n+1)B\right](i,j) \\ &= \sum_i \sum_j \sum_a [f_a \circ nB](i,j) - \sum_i \sum_j \sum_a [f_a \circ (n+1)B](i,j) \\ &= \sum_a \left[\sum_i \sum_j [f_a \circ nB - f_a \circ (n+1)B](i,j)\right] \\ &= \sum_{a=1}^{A} [PS(f_a)](n,B) \end{aligned} \tag{19}$$

An identical result to (19) is easily obtained for $n < 0$ by replacing openings $f \circ nB$ with closings $f \bullet nB$. Thus the pattern spectrum obeys the threshold-linear superposition. To illustrate this, consider the example of the 1-D image f in Section 3.2. Fixing $B = \{0,1\}$ yields

n	-2	-1	0	1	2	3	4	5	6
$[PS(f)](n)$	2	6	3	8	6	0	5	0	7
$[PS(f_4)](n)$	0	1	1	2	0	0	0	0	0
$[PS(f_3)](n)$	2	1	1	4	0	0	0	0	0
$[PS(f_2)](n)$	0	3	1	2	6	0	0	0	0
$[PS(f_1)](n)$	0	1	0	0	0	0	5	0	7

Computing the pattern spectra of the binary images f_a is much easier than for f. For example, for 1-D images f and $B = \{0,1\}$, the value of $PS(f_a)$ at $(n-1)$ is equal to n times the number of *runs* of n consecutive 1's if $n \geq 1$; likewise for runs of 0's and negative n.

Observe that, the pattern spectrum system performs an image measurement, because the system output $PS(f)$ is a two-parameter (n,B) function that measures the shape-size distribution of f. By contrast, all three previous morphological systems examined in Sections 3.1, 3.2, and 3.3 perform an image transformation because their output is another image signal.

4 General Result

The four morphological systems of Section 3, which we showed that obeyed the threshold-linear superposition, consisted of pointwise additions/subtractions of simple morphological operations. Next we show that these four examples are special cases of a more general result. Let F be the class of all real-valued *nonnegative signals* $f(x)$ (not necessarily images) with a d-dimensional (d = 1,2,...) argument x, continuous (i.e, real) or discrete (i.e., integer). Let S be the class of all systems $\Psi : F \rightarrow G$ that obey the threshold-linear superposition, with the restriction that either all $\Psi \in S$ are signal transformations or all are signal measurements but not both. G is the class of system outputs, which are either real-valued signals like the signals in F (but not necessarily nonnegative) or real-valued measurements (constants or functions of several parameters). Let us view each system Ψ in S as a vector point. Then let us define a binary operation $\Psi_1 + \Psi_2$ called *system (vector) addition* between any $\Psi_1, \Psi_2 \in S$ and a unary operation $r \cdot \Psi$ called *scalar multiplication of a system* Ψ by any real number $r \in \mathbf{R}$ as follows:

$$[\Psi_1 + \Psi_2](f) \stackrel{\text{def}}{=} \Psi_1(f) + \Psi_2(f) \quad , \quad \forall f \in F \tag{20}$$

$$[r \cdot \Psi](f) \stackrel{\text{def}}{=} r \cdot \Psi(f) \qquad , \quad \forall f \in F \tag{21}$$

There is a different interpretation of the symbols + and $\cdot$ between the left and right parts of these definitions. In the right part of (20) "+" denotes pointwise addition of signals if S is a class of signal transformations or addition of real numbers if S is a class of signal measurements. In the right part of (21) "$\cdot$" denotes multiplication of the signal or measurement $\Psi(f)$ by the scalar r. Thus $\Psi_1 + \Psi_2$ is a parallel interconnection of the systems Ψ_1 and Ψ_2, whereas $r \cdot \Psi$ just scales Ψ by r.

THEOREM 1 . *The class S of systems Ψ that obey the threshold-linear superposition forms a vector space over the field of real numbers under the vector addition (20) and scalar multiplication (21).*

Proof: From [23], we must prove that, for all $\Psi, \Phi \in S$ and $r, q \in \mathbf{R}$,

V1. $(S, +)$ is an Abelian group.
V2. $r \cdot \Psi \in S$.
V3. $r(\Psi + \Phi) = r \cdot \Psi + r \cdot \Phi$.
V4. $(r + q)\Psi = r \cdot \Psi + q \cdot \Psi$.
V5. $r \cdot (q \cdot \Psi) = (rq) \cdot \Psi$.
V6. $1 \cdot \Psi = \Psi$.

(V1): S is closed under system + because

$$[\Psi + \Phi](f) = \Psi(f) + \Phi(f) = \sum_a \Psi(f_a) + \sum_a \Phi(f_a) = \sum_a [\Phi + \Psi](f_a) \tag{22}$$

Further, the system + is associative, commutative, and has a *zero* element (the system Ψ_0, where $\Psi_0(f)$ is the all-zero signal for all f or just zero in case of signal measurements). Finally each Ψ has its *inverse* system $-\Psi$, defined as $[-\Psi](f) = -\Psi(f)$. Hence, $(S, +)$ is an Abelian group.

(V2) is true because

$$[r \cdot \Psi](f) = r \cdot \Psi(f) = r \cdot \sum_a \Psi(f_a) = \sum_a r \cdot \Psi(f_a) = \sum_a [r \cdot \Psi](f_a) \tag{23}$$

The proof of the rest of the axioms (V3)-(V6) is easy and hence omitted; it simply uses the results (22) and (23) together with elementary properties of the addition/multiplication on real numbers. Q.E.D.

The above result establishes that the principle of threshold-linear superposition is obeyed by any composite system formed as a linear combination of systems that obey it. As a special case consider systems Ψ_k among the following: erosion, dilation, rank-order filters, and cascade (e.g., openings, closings) or parallel (using pointwise max/min) combination of these. All such Ψ_k obey (8) as shown in [2,16,10,11]; hence, Theorem 1 implies that any linear combination system $\Psi(f) = \sum_k \Psi_k(f)$ will also obey (8). Therefore, the results for the four morphological systems of Section 3 follow now as simple corollaries of Theorem 1. Note also that the class of systems obeying threshold-linear superposition contains the class of all linear systems, because threshold-linear superposition is a weak form of linear superposition.

5 Concluding Remarks

An important factor on which our results in Section 3 depend is that the erosions, dilations, openings and closings used by the four analyzed morphological systems involve flat (binary) structuring elements. That is, for a 2-D image signal, only 2-D or 1-D sets can be used as structuring elements; likewise, for a 1-D signal, the structuring element must be a 1-D set. For the more general erosions (min of differences), dilations (max of sums), and their combinations, which use a non-binary structuring element [7,2,10,14], our results in this paper do not apply.

Although our analysis in Sections 2 and 3 refers to image signals, all the concepts and results are also valid for nonnegative input signals of any dimensionality. Likewise, the validity of the general theorem in Section 4 depends neither on the dimensionality of input signals nor on whether they have continuous or discrete argument. It only requires that the *input* signals (but not necessarily the outputs) be nonnegative. Hence it especially applies to image analysis systems.

The key idea of our results is that a large class of morphological and other system for graytone image analysis reduces to corresponding systems for binary signals. But the latter are much easier to analyze. Hence our results provide a theoretical tool that facilitates the analysis of many morphological and related nonlinear systems. In addition, they suggest new implementations based on threshold superposition. Of course, *software* implementations of these ideas on current serial computer architectures are discouraging because of the large number of thresholded binary images required. However, VLSI *hardware* implementations exploiting the threshold superposition of composite graytone morphological systems (as already has been done for simple rank-order and morphological filters [18]-[20]) is very promising because binary morphological operations can be done using only pixel counting. Further, the binary operations on each threshold binary image can be done in *parallel* for all threshold levels.

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