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# Conditions for Positivity of an Energy Operator

Alan C. Bovik and Petros Maragos

***Abstract*—We present necessary and sufficient conditions such that the output from the Teager-Kaiser energy operator $[\dot{s}(t) = ds(t)/dt]$**

$$\Psi_c[s(t)] = \dot{s}^2(t) - s(t)\ddot{s}(t) \tag{1}$$

**for continuous-time signals $s(t)$ and the output from the corresponding discrete-time energy operator**

$$\Psi_d[s(n)] = s^2(n) - s(n+1)s(n-1) \tag{2}$$

**be non-negative everywhere.**

## I. Introduction

The nonlinear signal operators in (1) and (2) were developed by Teager [1] in his work on speech modeling and were introduced recently by Kaiser [2], [3]. These operators have been shown to be effective for AM and FM demodulation in several useful classes of signals, such as speech and image signals [4]–[10]. $\Psi_c$ owes its energy-tracking capability to the fact that when it is applied to the output signal from a simple harmonic oscillator, it tracks the energy of the source generating the signal. A more general and particularly useful property of the energy operators (1), (2) are their behavior when applied to AM-FM signals of the form

$$s(t) = a(t)\cos[\phi(t)] \tag{3}$$

in the continuous case, and

$$s(n) = a(n)\cos[\phi(n)] \tag{4}$$

in the discrete case. Here we have

$$\Psi_c[s(t)] \approx a^2(t)\omega_i^2(t) \tag{5}$$

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the squared product of the amplitude $a(t)$ and the time-varying instantaneous frequency $\omega_i(t) = \dot{\phi}(t)$. Similarly in the discrete case

$$\Psi_d[s(n)] \approx a^2(n)\sin^2[\Omega_i(n)] \tag{6}$$

where $\Omega_i(n) \equiv d\phi(n)/dn$.

The approximations (5) and (6) hold quite well under very useful conditions expressed in terms of the smoothness or bandlimitedness of the amplitude modulation functions and the instantaneous frequencies. Detailed analyses are presented in [4]–[9], which show that the relative error is quite small for realistic signals in speech and other communications applications. These observations have led to the development of *energy separation algorithms* [5], [7], which attempt to separate the amplitude and frequency modulations in the products (5), (6) as distinct useful pieces of information.

The positivity of the energy operator output is a desired property for at least three fundamental reasons: (1) the interpretation of the output as some (normalized) physical energy; (2) the positive nature of the approximations in (5) and (6) needed for AM and/or FM demodulation [4], [6]; (3) the fact that the energy separation algorithms [5], [7] operate under this positivity assumption. In [6], [7] several sufficient conditions have been developed for the positivity of the energy operators. For example, $\Psi_c[s(t)] \geq 0$ if the signal $s(t)$ is any finite product of cosines, real exponentials, and linear trends. Alternatively, $\Psi_c[s(t)\ 0$ if $s(t)$ is an AM-FM signal and the amounts of amplitude/frequency modulation are not excessively large and the bandwidths of the amplitude/frequency modulating signals are reasonably smaller than the carrier frequency.

In the current paper, we explore general conditions on the arbitrary signal $s$ such that $\Psi_c(s)$ or $\Psi_d(s)$ be non-negative over the entire domain of analysis. Some of these conditions are necessary and sufficient and have an interesting geometric meaning, since they are expressed in terms of the concavity of the logarithm of the signal magnitude. We treat the continuous and discrete cases separately.

## II. Continuous Case

Henceforth, we suppose that the signal $s: \mathbf{D} \to \mathbf{R}$ has finite second derivatives everywhere (and hence is continuous) on some arbitrary set $\mathbf{D} \subseteq \mathbf{R}$. Lemma 1 is a simple, easily-tested sufficient condition for non-negativity of $\Psi_c(s)$. Lemma 2 is used to prove Theorem 1, although it also supplies interesting conditions for the non-negativity of the operator $\Psi_c$ for the special case of nonzero signals.

*Lemma 1:* At any $t \in \mathbf{D}, \Psi_c[s(t)] \geq 0$ if any of the following conditions hold:

(a) $s(t) = 0$. (b) $\ddot{s}(t) = 0$. (c) $s(t) > 0$ and $\ddot{s}(t) < 0$. (d) $s(t) < 0$ and $\ddot{s}(t) > 0$.

*Proof:* If any of (a)–(d) is true, then $s(t)\ddot{s}(t) \leq 0 \Rightarrow \Psi_c[s(t)] \geq 0$. ♦

Thus, if $\mathbf{I} \subseteq \mathbf{D}$ is some interval whose endpoints are either zeroes or inflection points of $s(t)$, then $\Psi_c[s] \geq 0 \forall t \in \mathbf{I}$ if either $s$ is positive and concave or if $s$ is negative and convex in the interior of $\mathbf{I}$. The next lemma gives necessary and sufficient conditions for everywhere nonzero signals, which are only slightly more complicated.

*Lemma 2:* Suppose that $s(t) \neq 0 \forall t \in \mathbf{D}$. Then the following three statements are equivalent:

(a) $\Psi_c[s(t)] \geq 0 \forall t \in \mathbf{D}$. (b) $\log[s^2(t)]$ is concave on $\mathbf{D}$. (c) $\log|s(t)|$ is concave on $\mathbf{D}$.

*Proof:* Let $g(t) = \log[s^2(t)]$. Then (a) $\Leftrightarrow$ (b) since $g(t)$ is concave on $\mathbf{D} \Leftrightarrow \ddot{g} \leq 0 \forall t \in \mathbf{D} \Leftrightarrow 2(s(t)\ddot{s}(t) - \dot{s}^2(t)/s^2(t)) \leq$

$0 \,\forall t \in \mathbf{D} \Leftrightarrow \Psi_c[s(t)] \geq 0 \,\forall t \in \mathbf{D}$. Furthermore, (b) $\Leftrightarrow$ (c) since $g(t) = 2 \log |s(t)|$. ♦

By Lemma 1(a), if $s(t_0) = 0$ for some $t_0 \in \mathbf{D}$, then $\Psi_c[s(t_0)] \geq 0$. Therefore, we can use the zeroes of a signal as boundary points separating intervals over which the signal is of one sign; on these intervals, Lemma 2 applies. This forms the basis of the more general Theorem 1. In proving the main theorem, we will use the following notations for the subsets of $\mathbf{D}$ over which $s(t)$ is nonzero or zero

$$\mathbf{D}_N = \{t \in \mathbf{D}: s(t) \neq 0\} \tag{7}$$

$$\mathbf{D}_Z = \{t \in \mathbf{D}: s(t) = 0\}. \tag{8}$$

Clearly, since $s$ is continuous on $\mathbf{D}$, $\mathbf{D}_N$ is an open set;[1] therefore, $\mathbf{D}_N$ may be written as a countable union of disjoint open intervals $\mathbf{I}_k$

$$\mathbf{D}_N = \bigcup_k \mathbf{I}_k. \tag{9}$$

These observations are used to set conditions on more general signals $s(t)$ that may have zeroes on $\mathbf{D}$, as Theorem 1 states next.

*Theorem 1:* Let $s: \mathbf{D} \rightarrow \mathbf{R}$ be twice finite differentiable on $\mathbf{D}$ and $\mathbf{D}_N = \{t \in \mathbf{D}: s(t) \neq 0\}$. Then the following statements are equivalent:

(a) $\Psi_c[s(t)] \geq 0 \,\forall t \in \mathbf{D}$. (b) $\log[s^2(t)]$ is concave on every open subinterval of $\mathbf{D}_N$. (c) $\log |s(t)|$ is concave on every open subinterval of $\mathbf{D}_N$.

*Proof:* Assume that the sets $\mathbf{D}_Z$ and $\mathbf{D}_N$ are nonempty; else either Lemma 1 or Lemma 2 apply directly. By Lemma 1(a) we need only consider points in the open set $\mathbf{D}_N = \cup_k \mathbf{I}_k$ where the open intervals $\mathbf{I}_k = (a_k, b_k)$ are disjoint. Hence, any open interval $\mathbf{B} \subseteq \mathbf{D}_N$ also satisfies $\mathbf{B} \subseteq \mathbf{I}_m$ for some $m$. By Lemma 2, $\Psi_c[s(t)] \geq 0 \,\forall t \in \mathbf{B} \Leftrightarrow \log[s^2(t)]$ is concave on $\mathbf{B} \Leftrightarrow \log |s(t)|$ is concave on $\mathbf{B}$. Since this is true for every $\mathbf{B} \subseteq \mathbf{I}_m$ and any $\mathbf{I}_m \subseteq \mathbf{D}_N$ we have that (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c). ♦

The following corollary is really just a restatement of Theorem 1, hence requires no proof. Although the statement of the corollary is a little less precise than Theorem 1, it is also somewhat more intuitive.

*Corollary 1:* Let $s: \mathbf{D} \rightarrow \mathbf{R}$ be twice finite differentiable on $\mathbf{D}$. Then the following statements are equivalent:

(a) $\Psi_c[s(t)] \geq 0 \,\forall t \in \mathbf{D}$. (b) $\log[s^2(t)]$ is concave between every two consecutive zeroes of $s(t)$. (c) $\log |s(t)|$ is concave between every two consecutive zeroes of $s(t)$. ♦

The simplest cases satisfying the above conditions are the linear signals $s(t) = at + b$ where $\Psi_c[s(t)] = a^2$, the sinusoidal signals $s(t) = A \sin(\omega t + \varphi)$ with $\Psi_c[s(t)] = (A\omega)^2$, and the real exponentials $s(t) = e^{rt}$ with $\Psi_c[s(t)] = 0$. More generally, oscillatory signals $s(t)$ will everywhere have non-negative $\Psi_c(s)$ regardless of their average frequency of oscillation, provided that the signal amplitude is sufficiently "smooth" between every two consecutive zeroes. Here "smoothness" is expressed in terms of logarithmic concavity: the rate of change of the slope of the logarithmic signal must not change its sign between consecutive zeros.

## III. Discrete Case

There are results for the discrete Teager-Kaiser energy operator $\Psi_d[s(n)]$ that are quite similar to the continuous case; however, there are also important differences that arise from the discrete approximations used. We assume that the discrete signal $s: \mathbf{J} \rightarrow \mathbf{R}$ is defined on the integer interval $\mathbf{J} = /a, b/ \subseteq \mathbf{Z}$, where $-\infty \leq a < b \leq \infty$. We denote

$$s''(n) = s(n-1) - 2s(n) + s(n+1) \tag{10}$$

and say that $s(n)$ is *concave* on $\mathbf{J}$ if $s''(n) \leq 0 \,\forall n \in \mathbf{J}$, and $s(n)$ is *convex* on $\mathbf{J}$ if $s''(n) \geq 0 \,\forall n \in \mathbf{J}$. If $\mathbf{J}$ is either finite or one-sided infinite, we shall make the simplifying assumption that $s(a-1) = s(b+1) = 0$, in order to handle the forward and backward differences used in defining $s''(n)$.

*Lemma 3:* If $s(n_0) = 0$ for some $n_0 \in \mathbf{J}$, then $\Psi_d[s(n_0 \pm 1)] \geq 0$.

*Proof:* $\Psi_d[s(n_0 \pm 1)] = s^2(n_0 \pm 1) - s(n_0 + 1 \pm 1)s(n_0 - 1 \pm 1) = s^2(n_0 \pm 1) \geq 0$. ♦

*Lemma 4:* At any $n \in \mathbf{J}$, $\Psi_d[s(n)] \geq 0$ if either of the following two conditions holds:

(a) $s(n) \geq 0$ and $s''(n) \leq 0 \,\forall n \in \mathbf{J}$. (b) $s(n) \leq 0$ and $s''(n) \geq 0 \,\forall n \in \mathbf{J}$.

*Proof:* Note that for any signal $s(n)$ and at each $n$

$$\frac{[s(n-1) + s(n+1)]^2}{4} \geq s(n-1)s(n+1). \tag{11}$$

If (a) is true, then $0 \geq s(n-1) - 2s(n) + s(n+1)$, or

$$s(n) \geq \frac{s(n-1) + s(n+1)}{2} \geq 0. \tag{12}$$

Squaring (12) and substituting into (11) yields the desired result. If (b) is true, the inequality in (12) is reversed; squaring still yields the desired result. ♦

Thus, if a signal $s$ is everywhere of one sign, then it suffices that $s$ be either non-negative and concave or nonpositive and convex in order that $\Psi_d(s)$ be non-negative. The next lemma is likewise analogous to Lemma 2 of the continuous case.

*Lemma 5:* Suppose that either $s(n) > 0 \,\forall n \in \mathbf{J}$ or $s(n) < 0 \,\forall n \in \mathbf{J}$. Then the following three statements are equivalent:

(a) $\Psi_d[s(n)] \geq 0 \,\forall n \in \mathbf{J}$. (b) $\log[s^2(n)]$ is concave on $\mathbf{J}$. (c) $\log |s(n)|$ is concave on $\mathbf{J}$.

*Proof:* Let $g(n) = \log[s^2(n)]$. Then (a) $\Leftrightarrow$ (b) since $g''(n)$ is concave on $\mathbf{J} \Leftrightarrow g''(n) \leq 0 \,\forall n \in \mathbf{J} \Leftrightarrow \log|s(n-1)| - 2\log|s(n)| + \log|s(n+1)| \leq 0 \,\forall n \in \mathbf{J} \Leftrightarrow \log[(s(n-1)s(n+1)/s^2(n))] \leq 0 \Leftrightarrow \Psi_d[s(n)] \geq 0 \,\forall n \in \mathbf{J}$. (b) $\Leftrightarrow$ (c) since $\log[s^2(n)] = 2\log|s(n)|$. ♦

The final result supplies the most general necessary and sufficient condition on a discrete-time signal $s(n)$ such that $\Psi_d(s)$ be non-negative. The conditions in the statement of Theorem 2 are more complicated than in the continuous case, because of the peculiar behavior of the energy operator near the zeroes of a signal. As before we denote the sets

$$\mathbf{J}_N = \{n \in \mathbf{J}: s(n) \neq 0\} \tag{13}$$

$$\mathbf{J}_Z = \{n \in \mathbf{J}: s(n) = 0\}. \tag{14}$$

*Theorem 2:* The following statements are equivalent:

(a) $\Psi_d[s(n)] \geq 0 \,\forall n \in \mathbf{J}$. (b) $\log[s^2(n)]$ is concave on every integer subinterval of $\mathbf{J}$ on which $s(n)$ is constant (nonzero) sign and $s(n-1)s(n+1) \leq 0$ whenever $s(n) = 0$. (c) $\log|s(n)|$ is concave on every integer subinterval of $\mathbf{J}$ on which $s(n)$ is of constant (nonzero) sign and $s(n-1)s(n+1) \leq 0$ whenever $s(n) = 0$.

*Proof:* We assume that $\mathbf{J}_N$ and $\mathbf{J}_Z$ are nonempty; else Lemma 3 or 5 applies. Therefore, we may partition $\mathbf{J}_N$ into integer subintervals $\mathbf{Z}_k = /a_k, b_k/$ such that either $s(n) > 0 \,\forall n \in \mathbf{Z}_k$ or $s(n) < 0 \,\forall n \in \mathbf{Z}_k$. Suppose that (a) is true. Then for $n \in \mathbf{J}_Z$, $\Psi_d[s(n)] = -s(n-1)s(n+1) \geq 0$. Otherwise, $n \in \mathbf{Z}_k \subseteq \mathbf{J}_N$ for some $k$. But since for any $k$ we have that $\Psi_d[s(n)] \geq 0 \,\forall n \in \mathbf{Z}_k \Leftrightarrow \log[s^2(n)]$ is concave on $\mathbf{Z}_k \Leftrightarrow \log|s(n)|$ is concave on $\mathbf{Z}_k$, then (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c). Finally, (b) $\Rightarrow$ (a), trivially in the case $n \in \mathbf{J}_Z$ and by Lemma 5 in the case $n \in \mathbf{Z}_k \subseteq \mathbf{J}_N$. ♦

Again, simple examples satisfying the above conditions are the linear signals $s(n) = an + b$, discrete-time sinusoidal signals $s(n) = A \sin(\Omega n + \theta)$, and real exponentials $s(n) = r^n$.

[1] For any $t_0 \in \mathbf{D}_N$, the continuity of $s \Rightarrow \exists$ an open interval $\mathbf{I}_0 \subseteq \mathbf{D}_N$ containing $t_0 \Rightarrow \mathbf{D}_N$ is an open set.

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# A Critical Study of a Self-Calibrating Direction-Finding Method for Arrays

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***Abstract*—The self-calibrating direction-finding method was developed by Friedlander and Weiss. It refines the estimates of the array element gains and phases while it estimates the signal directions. This study shows that the outputs of the method could be inaccurate, even if noise is absent.**

## I. Introduction

This is a study of the self-calibrating direction-finding method developed by Friedlander and Weiss [1], [2] for nonlinear arrays. The method (the F-W method) uses the outputs of an array to refine the element gains and phases while it estimates the signal directions. It is based on an equation derived from the MUSIC method [3].

This study shows that the above equation does not always have unique solutions. The output estimates of the F-W method could therefore be inaccurate, even if noise is absent.

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## II. The Method

The assumptions made are those underlying the MUSIC method plus the following:

(a) The array is nonlinear.

(b) The number of signals $N$ and the number of array elements $M$ are related by $2 \leq N < M$.

In the signal model used, an array snapshot is given by

$$\mathbf{x} = (\mathbf{\Gamma}\ \mathbf{a}(\theta_1), \mathbf{\Gamma}\ \mathbf{a}(\theta_2), \ldots, \mathbf{\Gamma}\ \mathbf{a}(\theta_N))\mathbf{s} + \eta \tag{1}$$

where

$$\mathbf{\Gamma} = \mathrm{diag}\{g_1 \exp[j\psi_1], g_2 \exp[j\psi_2], \ldots, g_M \exp[j\psi_M]\}. \tag{2}$$

The set $\{g_m : m = 1, 2, \ldots, M\}$ denotes the element gains, $\{\psi_m : m = 1, 2, \ldots, M\}$ denotes the element phases, $\{\theta_n : n = 1, 2, \ldots, N\}$ denotes the signal directions, $\mathbf{a}(\theta)$ is an array steering vector for direction $\theta$, $\mathbf{s}$ is the complex signal amplitudes at a reference array element, and $\eta$ is a noise vector. The uniqueness condition for the element gains and phases is $\{g_1 = 1, \psi_1 = 0\}$.

The method is based on the observation that each $\mathbf{\Gamma}\ \mathbf{a}(\theta_\mathbf{n})$ is a vector in the signal space and is orthogonal to every vector in the noise space so that

$$\mathbf{U}^H \mathbf{\Gamma}\ \mathbf{a}(\theta_n) = \mathbf{O}_{M-N}, \qquad n = 1, 2, \ldots, N, \tag{3}$$

where $\mathbf{U}$ is an $M \times (M-N)$ matrix constructed with a set of $M$-$N$ orthonormal basis vectors for the noise subspace, and $\mathbf{O_{M-N}}$ is a null vector with $M$-$N$ components. Let

$$C(\mathbf{U}, \mathbf{\Gamma}', \theta_n') \Delta \sum_{n=1}^{N} \|\mathbf{U}^H \mathbf{\Gamma}' \mathbf{a}(\theta_n')\|^2 \tag{4}$$

be a cost function defined in terms of $\mathbf{U}$, an arbitrary $\mathbf{\Gamma}'$, and a set of arbitrary directions $\{\theta_n' : n = 1, 2, \ldots, N\}$. From (3), the equation

$$C(\mathbf{U}, \mathbf{\Gamma}', \theta_n') = 0 \tag{5}$$

has a set of solutions given by $\{\mathbf{\Gamma}, \{\theta_\mathbf{n}\}\}$, i.e., the set comprising $\mathbf{\Gamma}$ and $\{\theta_n : n = 1, 2, \ldots, N\}$

In later discussions, the set $\{\mathbf{\Gamma}, \{\theta_\mathbf{n}\}\}$ is sometimes treated as the location of a zero or a minimum in the cost function (4).

The procedure for calculating the estimates $\hat{\mathbf{\Gamma}}$ and $\{\hat{\theta}_\mathbf{n} : n = 1, 2, \ldots, N\}$ is described in [1] and [2]. Its important features are as follows:

Step 1. Construct $\hat{\mathbf{U}}$, an estimate of $\mathbf{U}$.

Step 2. Set $k = 0$; set $\mathbf{\Gamma^{(k)}} = \mathbf{\Gamma_0}$, where $\mathbf{\Gamma_0}$ is based on nominal values or some recent calibration information.

Step 3. Identify the $N$ highest peaks in the MUSIC spectrum

$$P\left(\theta | \mathbf{\Gamma}^{(k)}\right) = + - \|\hat{\mathbf{U}}^H \mathbf{\Gamma}^{(k)} \mathbf{a}(\theta)\|^{-2} \tag{6}$$

as $\left\{\theta_n^{(k)} : n = 1, 2, \ldots, N\right\}$.

Step 4. Find the $\mathbf{\Gamma}'$ that minimizes $C\left(\hat{\mathbf{U}}, \mathbf{\Gamma}', \theta_\mathbf{n}^{(\mathbf{k})}\right)$, a cost function constructed with $\hat{\mathbf{U}}$ and $\{\theta_n^{(k)}\}$. Denote it by $\mathbf{\Gamma^{(k+1)}}$ and the corresponding minimum cost by $J_k$.

Step 5. If $J_{k-1} - J_k > \varepsilon$, a preset threshold, then $k = k + 1$; return to Step 3. Else, identify the output estimates as $\hat{\mathbf{\Gamma}} = \mathbf{\Gamma^{(k+1)}}$ and $\{\hat{\theta}_n\} = \{\theta_n^{(k)}\}$.

For convenience, $\{\theta_n^{(k)}\}$, say, has been used to denote $\{\theta_n^{(k)} : n = 1, 2, \ldots, N\}$.