Chapter 1

Partial Differential Equations for Morphological Operators

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1.1 Introduction

Two of G. Matheron's seminal contributions have been his development of size distributions (else called 'granulometries') and his kernel representation theory. The first deals with semigroups of multiscale openings and closings of binary images (shapes) by compact convex sets, a basic ingredient of which are the multiscale Minkowski dilations and erosions. The second deals with representing increasing and translation-invariant set operators as union of erosions by its kernel sets or as an intersection of dilations.

The semigroup structure of the basic multiscale morphological operators led to the development (by Alvarez et al. [2], Brockett & Maragos [9], and Boomgaard & Smeulders [60]) of Partial Differential Equations (PDEs) that can generate them on a continuum of scales. In parallel, the representation theory was extended by Maragos [36] to function operators as sup-inf of min-max filterings by elements of a kernel basis. These two seemingly unrelated research directions were later rejoined by Catte et al. [11] and by Guichard & Morel [22, 23] who used the basis representation of multiscale sup-inf operators to develop PDEs that can generate them based on variants of the mean curvature motion.

Many information extraction tasks in image processing and computer vision necessitate the analysis at multiple scales. Influenced by the work of Marr (and coworkers) [42], Koenderink [31] and Witkin [63], for more than a decade the multiscale analysis was based on Gaussian convolutions. The popularity of this approach was due to its linearity and its relationship to the linear isotropic heat diffusion PDE. The big disadvantage of the Gaussian scale-space approach is the fact that linear smoothers blur and shift important image features, e.g., edges. There is, however, a variety of *nonlinear* smoothing filters, including morphological open-closings (of the Minkowski type [43, 56] or of the reconstruction [53] and leveling type [47, 39]) and anisotropic nonlinear diffusion [51], which can smooth while preserving important image features and can provide a nonlinear scale-space.

Until the end of the 1990s, morphological image processing had been based traditionally on modelling images as sets or as points in a complete lattice of functions and viewing morphological image transformations as set or lattice operators. Further, the vast majority of implementations of multiscale morphological filtering had been discrete. In 1992, inspired by the modelling of the Gaussian scale-space via the linear heat diffusion PDE, three teams of researchers independently published nonlinear PDEs that model the continuous multiscale morphological scale-space. Specifically, Alvarez, Guichard, Lions and Morel [1] obtained PDEs for multiscale flat dilation and erosion, by compact convex structuring sets, as part of their general work on developing PDE-based models for multiscale image processing that satisfy certain axiomatic principles. Brockett and Maragos [8] developed PDEs that model multiscale morphological dilation, erosion, opening and closing by compact-support structuring elements that are either convex sets or concave functions and may have non-smooth boundaries or graphs, respectively. Their work was based on the semigroup structure of the multiscale dilation and erosion operators and the use of morphological sup/inf derivatives to deal with the development of shocks (i.e., discontinuities in the derivatives).

In [59, Ch. 8], Boomgaard and Smeulders obtained PDEs for multiscale dilation and erosion by studying the propagation of the boundaries of 2D sets and the graphs of signals under multiscale dilation and erosion. Their work applies to convex structuring elements whose boundaries contain no linear segments, are smooth and possess a unique normal at each point. Refinements of the above three works for PDEs modelling multiscale morphology followed in [2, 3, 9, 38, 40, 60]. Extensions also followed in several directions including asymptotic analysis and iterated filtering by Guichard & Morel [22, 23], a unification of morphological PDEs using Legendre-Fenchel 'slope' transforms by Heijmans & Maragos [25], a common algebraic framework for linear and morphological scale-spaces by Heijmans & Boomgaard [26] and PDEs for morphological reconstruction operators with global constraints by Maragos and Meyer [47, 39].

To illustrate the basic idea behind morphological PDEs, we consider a 1D example, for which we define the multiscale flat dilation and erosion of a 1D signal $f(x)$ by the set $[-t,t]$ as the scale-space functions

$$\delta(x,t) = \sup_{|y|\leq t} f(x-y), \qquad \varepsilon(x,t) = \inf_{|y|\leq t} f(x+y).$$

The PDEs generating these multiscale flat dilations and erosions are [9]

$$\begin{aligned} \partial\delta/\partial t &= |\partial\delta/\partial x|, \qquad \partial\varepsilon/\partial t = -|\partial\varepsilon/\partial x|, \\ \delta(x,0) &= \varepsilon(x,0) = f(x). \end{aligned} \tag{1.1}$$

In parallel to the development of the above ideas, there have been some advances in the field of differential geometry for evolving curves or surfaces using level set methods. Specifically, Osher & Sethian [50] have developed PDEs of the Hamilton-Jacobi type to model the propagation of curves, embedded as level curves (isoheight contours) of functions evolving in scale-space. The propagation was modelled using speeds along directions normal to the curve that contain a constant term and/or a term dependent on curvature. Furthermore, they developed robust numerical algorithms to solve these PDEs by using stable and shock-capturing schemes to solve similar, shock-producing, nonlinear wave PDEs that are related to hyperbolic conservation laws [32]. Kimia, Zucker & Tannenbaum [29] have applied and extended these curve evolution ideas to shape analysis in computer vision. Arehart, Vincent & Kimia [4] and Sapiro et al. [54] implemented continuous-scale morphological dilations and erosions using the numerical algorithms of curve evolution to solve the PDEs for multiscale dilation and erosion. There are several relationships between curve evolution and multiscale morphology, since the evolution with constant normal speed models multiscale set dilation, and the corresponding Hamilton-Jacobi PDEs contain the PDE of multiscale dilation/erosion by disks as a basic ingredient. Furthermore, the level sets used in curve evolution have previously been used extensively in mathematical morphology for extending set operations to functions [56], [41].

Multiscale dilations and erosions of binary images can also be obtained via distance transforms. Using Huygens' construction, the boundaries of multiscale dilations–erosions by disks can also be viewed as the wavefronts of a wave initiating from the original image boundary and propagating with constant normal speed in a homogeneous medium [7]. This idea can also be extended to heterogeneous media by using a weighted distance function, in which the weights are inversely proportional to the propagation speeds. In geometrical optics, these distance wavefronts are obtained from the isolevel contours of the solution of the Eikonal PDE. This ubiquitous PDE (or its solution as weighted distance) has been applied to solving various problems in image analysis and computer vision [27] such as shape-from-shading [52, 30], gridless halftoning, and image segmentation [61, 46, 49, 38, 40].

Modelling linear and morphological scale-space analysis via PDEs has several advantages, mathematical, physical, and computational. In particular, there exist several efficient numerical algorithms which implement morphology-related PDEs on a discrete grid [50, 58, 23]. Thus, one can have both the advantages of continuous modelling and discrete processing.

This chapter is organized as follows. In section 1.2, we review all first-order PDEs coming from the asymptotic form of classical multiscale dilations and erosions. In section 1.3, we state the most general results about PDEs associated with the rescaling of any local increasing operator. Section 1.4 treats the opposite viewpoint : instead of constructing the PDE by iterating local morphological operators, it starts with a *scale space* abstract set of axioms on multiscale image analysis. A scale space in this abstract setting is nothing but a scale indexed family of operators T_t, understood as operators smoothing more and more the image when the scale t increases. Under sound axioms, it can be proved that scale spaces are equivalent to the action of nonlinear or linear parabolic PDEs. A further classification of the PDEs is sketched, according to their invariance properties. Section 1.5 takes the last turn by focusing on the curve evolution interpretation of all that. Actually, all contrast invariant image scale spaces can be described as curve scale spaces applied to each level line of the image. This point of view has become popular under the name of "level set methods" and yields the nice geometric interpretation of contrast invariant scale spaces as "curvature flows". Needless to be said, this rich subject cannot be but sketched in one book chapter and actually deserves a long and mathematically clean presentation. Probably the presentations closest to our viewpoint here are F. Cao's book [10] and the book to appear [23].

1.2 PDEs for Multiscale Morphological Operators

The main tools of low-level morphological image processing are a broad class of nonlinear signal operators formed as parallel and/or serial interconnections of the two most elementary morphological signal operators, the Minkowski *dilation* $\oplus$ and the *erosion* $\ominus$:

$$
\begin{aligned}
(f \oplus g)(x) &\triangleq \bigvee_{y \in \mathbb{E}} f(y) + g(x-y) \\
(f \ominus g)(x) &\triangleq \bigwedge_{y \in \mathbb{E}} f(y) - g(y-x),
\end{aligned}
$$

where $\bigvee$ and $\bigwedge$ denote supremum and infimum, and the signal domain can be continuous $\mathbb{E} = \mathbb{R}^d$ or discrete $\mathbb{E} = \mathbb{Z}^d$. The signal range is a subset of $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. Compositions of erosions and dilations yield two useful smoothing filters: the *opening* $f \mapsto (f \ominus g) \oplus g$ and *closing* $f \mapsto (f \oplus g) \ominus g$.

1.2.1 PDEs Generating Dilations and Erosions

Let $k : \mathbb{R}^2 \to \overline{\mathbb{R}}$ be a unit-scale upper-semicontinuous concave structuring function, to be used as the kernel for morphological dilations and erosions. Scaling both its values and its support by a scale parameter $t \geq 0$ yields a parameterized family of multiscale structuring functions

$$
k_t(x,y) \triangleq \begin{cases} tk(x/t, y/t), & \text{for } t > 0, \\ 0 \text{ at } (x,y) = (0,0) \text{ and } -\infty \text{ else}, & \text{for } t = 0, \end{cases} \tag{1.2}
$$

which satisfies the semigroup property

$$
k_s \oplus k_t = k_{s+t}. \tag{1.3}
$$

Using k_t in place of g as the kernel in the basic morphological operations leads to defining the *multiscale* dilation and erosion of $f : \mathbb{R}^2 \to \mathbb{R}$ by k_t as the scale-space functions

$$
\delta(x,y,t) \triangleq f \oplus k_t(x,y), \qquad \varepsilon(x,y,t) \triangleq f \ominus k_t(x,y), \tag{1.4}
$$

where $\delta(x,y,0) = \varepsilon(x,y,0) = f(x,y)$.

In practice, a useful class of functions k consists of flat structuring functions

$$k(x,y) = \begin{cases} 0 & \text{for } (x,y) \in B, \\ -\infty & \text{for } (x,y) \notin B, \end{cases} \tag{1.5}$$

which are the $0/-\infty$ indicator functions of compact convex planar sets B. The general PDE[1] generating the multiscale flat dilations of f by a general compact convex symmetric B is [2, 9, 25]

$$\frac{\partial \delta}{\partial t} = \mathrm{sptf}_B(\delta_x, \delta_y), \tag{1.6}$$

where $\mathrm{sptf}_B(\cdot)$ is the support function of B:

$$\mathrm{sptf}_B(x,y) \triangleq \bigvee_{(a,b)\in B} ax + by. \tag{1.7}$$

Useful cases of structuring sets B are obtained by the unit balls $B_p = \{(x,y) : \|(x,y)\|_p \leq 1\}$ of the metrics induced by the L_p norms $\|\cdot\|_p$, for $p = 1, 2, \ldots, \infty$. The PDEs generating the multiscale flat dilations of f by B_p for three special cases of norms $\|\cdot\|_p$ are as follows:

$$B = \text{rhombus } (p=1) \implies \delta_t = \max(|\delta_x|, |\delta_y|) = \|\nabla\delta\|_\infty, \tag{1.8}$$

$$B = \text{disk } (p=2) \implies \delta_t = \sqrt{(\delta_x)^2 + (\delta_y)^2} = \|\nabla\delta\|_2, \tag{1.9}$$

$$B = \text{square } (p=\infty) \implies \delta_t = |\delta_x| + |\delta_y| = \|\nabla\delta\|_1, \tag{1.10}$$

with $\delta(x,y,0) = f(x,y)$. The corresponding PDEs generating mutliscale flat erosions are

$$B = \text{rhombus} \implies \varepsilon_t = -\|\nabla\varepsilon\|_\infty, \tag{1.11}$$

$$B = \text{disk} \implies \varepsilon_t = -\|\nabla\varepsilon\|_2, \tag{1.12}$$

$$B = \text{square} \implies \varepsilon_t = -\|\nabla\varepsilon\|_1, \tag{1.13}$$

with $\varepsilon(x,y,0) = f(x,y)$.

These simple but nonlinear PDEs are satisfied at points where the data are smooth, that is, the partial derivatives exist. However, even if the initial image or signal f is smooth, at finite scales $t > 0$ the above dilation or erosion evolution may create discontinuities in the derivatives, called *shocks*,

[1] Notation often used for PDEs: $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$, $Du = \nabla u = (u_x, u_y)$, $\mathrm{div}(v,w)) = \nabla \cdot (v,w) = v_x + w_y$.

which then continue propagating in scale-space. Thus, the multiscale dilations δ or erosions ε are *weak solutions* of the corresponding PDEs, in the sense put forth by Lax [32]. Ways to deal with these shocks include replacing standard derivatives with morphological derivatives [9] or replacing the PDEs with differential inclusions [44]. The most acknowledged viewpoint on this, however, is to use the concept of viscosity solutions. For first-order PDEs, a good exposition is given in Barles [5] or in the classic [14]. Probably the shortest, more pedagogic and up to date presentation of viscosity solutions is the recent one by Crandall [13].

Next, we provide two examples of PDEs generating multiscale dilations by *graylevel structuring functions*. First, if we use the compact-support spherical function

$$k(x,y) = \begin{cases} \sqrt{1+x^2+y^2} & \text{for } x^2+y^2 \leq 1, \\ -\infty & \text{for } x^2+y^2 > 1, \end{cases} \tag{1.14}$$

the dilation PDE becomes

$$\delta_t = \sqrt{1+(\delta_x)^2+(\delta_y)^2}. \tag{1.15}$$

As shown in [9], this can be proven by using the semigroup structure of dilations and the first-order Taylor's approximation for the difference between dilations at scales t and $t+dt$. Alternatively, it can be proven using slope transforms, as explained in the next section. As a second example of structuring function, if k is the infinite-support parabola

$$k(x,y) = -r(x^2+y^2), \quad r > 0, \tag{1.16}$$

the dilation PDE becomes

$$\delta_t = [(\delta_x)^2 + (\delta_y)^2]/4r. \tag{1.17}$$

This can be proven using slope transforms.

1.2.2 Slope Transforms and Dilation PDEs

All of the above dilation (and erosion) PDEs can be unified using slope transforms. These transforms [37, 15] are simple variations of the Legendre-Fenchel transform. The word 'slope' was given only for insights because the eigenfunctions of a morphological dilation-erosion system are straight lines parameterized by their slope. Further, for morphological systems we can consider a new domain, called a 'slope domain', where morphological sup-inf convolutions in the time-space domain become addition of slope transforms in the slope domain.

Let the unit-scale kernel $k(x, y)$ be a general upper-semicontinuous concave function and consider its upper slope transform[2]

$$K^{\vee}(a, b) \triangleq \bigvee_{(x,y)\in\mathbb{R}^2} k(x, y) - (ax + by) \tag{1.18}$$

Then, as shown in [25, 44], the PDE generating multiscale signal dilations by k is

$$\partial\delta/\partial t = K^{\vee}(\delta_x, \delta_y) \tag{1.19}$$

Thus, the rate of change of δ in the scale (t) direction is equal to the upper slope transform of the structuring function evaluated at the spatial gradient of δ. Similarly, the PDE generating the multiscale erosion by k is

$$\partial\varepsilon/\partial t = -K^{\vee}(\varepsilon_x, \varepsilon_y). \tag{1.20}$$

For example, the PDE (1.6) modelling the general flat dilation by a compact convex set B is a special case of (1.19) since the support function (1.7) of B is the upper slope transform of the $0/-\infty$ indicator function of B. Likewise, the PDE (1.17) modelling multiscale dilations by parabolae results simply from (1.19) by noting that the upper slope transform of a concave parabola is a convex parabola.

All of the dilation and erosion PDEs examined are special cases of Hamilton-Jacobi equations, which are of paramount importance in physics. Such equations usually do not admit classic (i.e., everywhere differentiable) solutions. Viscosity solutions of Hamilton-Jacobi PDEs have been extensively studied by Crandall et al. [14]. The theory of viscosity solutions has been applied to morphological PDEs by Guichard & Morel [23]. Finally, Heijmans & Maragos [25] have shown via slope transforms that the multiscale dilation by a general upper-semicontinuous concave function is the viscosity solution of the Hamilton-Jacobi dilation PDE of Eq. (1.19).

1.2.3 PDEs Generating Openings and Closings

Let $u(x, y, t) = [f(x, y) \ominus tB] \oplus tB$ be the multiscale flat opening of an image f by the disk B. This standard opening can be generated at any scale $r > 0$

[2]In convex analysis, given a convex function $h(x)$ there uniquely corresponds another convex function $h^*(a) = \bigvee_x a \cdot x - h(x)$, called the *Legendre-Fenchel conjugate* of h. The lower slope transform of h, defined as $H^{\wedge}(a) = \bigwedge_x h(x) - a \cdot x$, is the dual of the upper slope transform. Obviously, the former is closely related to the conjugate function since $h^*(a) = -H^{\wedge}(a)$.

by running the following PDE [2]

$$u_t = -\max\left(\mathrm{sgn}(r-t), 0\right)\|\nabla u\|_2 + \max\left(\mathrm{sgn}(t-r), 0\right)\|\nabla u\|_2, \qquad (1.21)$$

from time $t = 0$ until time $t = 2r$ with initial condition $u(x, y, 0) = f(x, y)$, where $\mathrm{sgn}(\cdot)$ denotes the signum function. This PDE has time-dependent switching coefficients that make it act as an erosion PDE during $t \in [0, r]$ but as a dilation PDE during $t \in [r, 2r]$. At the switching instant $t = r$ this PDE exhibits discontinuities. This can be dealt with by making appropriate changes to the time scale that make time 'slow down' when approaching the discontinuity at $t = r$, as suggested by Alvarez et al. [2]. Of course, the solution u of the above PDE is an opening only at time $t = r$, whereas the solutions at other times is not a opening. In a different work, Brockett & Maragos [9] have developed a partial differential-difference equation that models at all times the evolutions of multiscale openings of 1D images by flat intervals. This does not involve only local operations but also global features such as the support geometry of peaks of f at various scales.

The reconstruction openings have found many more applications than the standard openings in a large variety of problems. We next present a nonlinear PDE that can model and generate openings and closings by reconstruction. Consider a 2D reference signal $f(x, y)$ and a marker signal $g(x, y)$. If $g \leq f$ everywhere and we start iteratively growing g via incremental flat dilations with an infinitesimally small disk ΔtB but without ever growing the result above the graph of f, then in the limit we shall have produced the reconstruction opening of f (with respect to the marker g). The infinitesimal generator of this signal evolution $u(x, y, t)$ can be modelled via the following PDE, studied by by Maragos & Meyer [47, 39],

$$\begin{aligned} u_t(x, y, t) &= \|\nabla u\|\mathrm{sgn}[f(x, y) - u(x, y, t)], \\ u(x, y, 0) &= g(x, y), \end{aligned} \qquad (1.22)$$

where $\mathrm{sgn}(r)$ equals 1 if $r > 0$, -1 if $r < 0$ and 0 if $r = 0$. The mapping from the initial value $u_0(x, y) = u(x, y, 0)$ to the limit $u_\infty(x, y) = \lim_{t\to\infty} u(x, y, t)$ is the *reconstruction opening* filter. If we reverse the roles of f and g, in the limit we obtain the *reconstruction closing* of f with respect to the marker g. Now, if there is no specific order between f and g, the PDE has a sign-varying coefficient $\mathrm{sgn}(f - u)$ with spatiotemporal dependence, which acts as a global constraint that controls the instantaneous growth. The final result $u_\infty(x, y)$ is equal to the output from a more general class of morphological filters, called *levelings* [47], which have many useful scale-space properties and contain as special cases the reconstruction openings and closings. For stability of the solution of the leveling PDE, g has to be uniformly continuous in the viscosity sense.

1.3 Asymptotic of Increasing Operators

We consider a family $\mathcal{F}$ of functions from $\mathbb{E}$ into $\mathbb{R}$ representing a class of images. An operator S, from $\mathcal{F}$ into $\mathcal{F}$, is said **increasing** or **monotone** if $\forall f, g \in \mathcal{F}$, $(\forall \mathbf{x} \in \mathbb{E}, f(\mathbf{x}) \geq g(\mathbf{x})) \Longrightarrow (\forall \mathbf{x}, Sf(\mathbf{x}) \geq Sg(\mathbf{x}))$.

In all the following we will assume that S commutes with spatial translations of the image, in other words we assume that S is invariant by translation.

Note: It is a general property of the increasing and translation invariant operators to preserve the Lipschitz property of any Lipschitz function. Consequently, a possible choice for $\mathcal{F}$ can be made by considering the set of Lipschitz functions.

1.3.1 Increasing Operators

The following formulae, inspired from work of Matheron [43], Serra [56], and Maragos [36] gives us a general form for any increasing and translation invariant operator:

Let S be a increasing function operator defined of $\mathcal{F}$, invariant by translation and commuting with the addition of constants. There exists a family $\mathbb{F}_1(S)$ of functions from $\mathbb{E}$ into $\mathbb{R} \cup \{-\infty, +\infty\}$ such that for all functions f of $\mathcal{F}$, we have

$$Sf(\mathbf{x}) = \bigwedge_{g \in \mathbb{F}_1(S)} \bigvee_{\mathbf{y} \in \mathbb{E}} f(\mathbf{y}) - g(\mathbf{x} - \mathbf{y}).$$

Similarly, there exists another family of functions $\mathbb{F}_2(S)$ such that

$$Sf(\mathbf{x}) = \bigvee_{g \in \mathbb{F}_2(S)} \bigwedge_{\mathbf{y} \in \mathbb{E}} f(\mathbf{y}) - g(\mathbf{x} - \mathbf{y}).$$

The special cases where $\mathbb{F}$ are made of a single function g correspond to the classical Minkowski dilation and erosion that have already been presented in section 1.2.

Examples of classical increasing operators (or "filters") that cannot be represented with a $\mathbb{F}$ made of a single function are e.g. the **"median"** filter or the **"mean"** filter. In fact, it would be probably vain to try to classify all possible increasing filters. So, in this section, we wish to specify the general forms of the PDEs related to increasing filters.

1.3.2 Scaled and Local Increasing Operators

We consider a scaled increasing operator S_h, where the scale h is a positive real number. We say that S_h is a **local increasing operator** if for all u and v such that $u(\mathbf{y}) > v(\mathbf{y})$ for $\mathbf{y}$ in a neighborhood of $\mathbf{x}$ and $\mathbf{y} \neq \mathbf{x}$, then for h small enough we have

$$(S_h u)(\mathbf{x}) \geq (S_h v)(\mathbf{x})$$

Roughly speaking, a local increasing operator is a scale operator whose action is reduced when its scale decreases. Easy way to construct a local increasing operator S_h from an increasing operator S is to localize the action of the family of functions $I\!F$: e.g., one can set S_h as in [24]:

$$S_h(u)(\mathbf{x}) = \bigwedge_{g \in I\!F} \bigvee_{\mathbf{y} \in \mathbb{E}} (u(\mathbf{x}+\mathbf{y}) - h^\beta g(\mathbf{y}/h^\alpha)), \tag{1.23}$$

for some $\alpha, \beta \geq 0$. This construction, with adequate choices of α and β will transform e.g. the mean, median, erosion or dilation filters on a disk of radius 1, into their corresponding respective localized versions on a radius h disk. However, in general, this construction is not sufficient to get a local increasing operator from any increasing operator S.

We finally need some technical assumption stating that a very smooth image must evolve in a smooth way with the considered operator. Let us recall that we initially assume that the operator is translation-invariant, so that the analysis on its asymptotic could be done at $\mathbf{x}$=0 or any other point $\mathbf{x}$. So choosing any point $\mathbf{x}$, let $Q_{A,p,c}(\mathbf{y}) = \frac{1}{2}(A(\mathbf{y}-\mathbf{x}), \mathbf{y}-\mathbf{x}) + (p, \mathbf{y}-\mathbf{x}) + c$ be a quadratic form on $\mathbb{E}$. (If $\mathbb{E} = \mathbb{R}^N$ then A is a $N * N$ matrix ($A = D^2Q(\mathbf{x})$), p a vector of $\mathbb{R}^N$ ($p = DQ(\mathbf{x})$) and c a constant.)

We shall say that a local increasing operator is **regular** if there exists a function $F(A, p, c)$, continuous with respect to A, such that

$$\forall Q_{A,p,c}, \quad \frac{(S_h Q - Q)(\mathbf{x})}{h} \to F(A, p, c) \quad \text{when} \quad h \to 0.$$

In [2], Alvarez et al gave the general asymptotic shape of any local and increasing operator:

Fundamental Asymptotic Theorem: *Let S_h be a local regular increasing operator and F the real function associated with the regularity assumption. Then S_h satisfies*

$$((S_h u - u)/h)(\mathbf{x}) \to F(D^2u(\mathbf{x}), Du(\mathbf{x}), u(\mathbf{x})) \tag{1.24}$$

as h tends to 0^+ for every C^2 function u and every $\mathbf{x}$. In addition, F is nondecreasing with respect to its first argument : If $A \geq \tilde{A}$, for the ordering of symmetric matrices,

$$\textit{then,} \quad F(A,p,c) \geq F(\tilde{A},p,c)). \tag{1.25}$$

This easy to prove theorem reduces the classification of all iterated local and increasing operators to the classification of all interesting functions F. In dimension 2, these real functions have six arguments. This number, however, can be drastically reduced when we impose obvious and rather necessary and useful invariance properties to the increasing operator.

This theorem also shows that the study of the asymptotic behavior of an increasing operator can be reduced to the study of its action on a parabolic function ($Q_{A,p,c}$).

1.4 The Scale-Space Framework

In this section, we consider an abstract framework, the "scale space", which at the end boils down, from the algorithmic viewpoint, to iterated filtering. Now, this framework will make it easier to classify and model the possible asymptotic behaviors of iterated increasing operators.

The scale space theory was founded (in a linear framework) by Witkin [63], Marr [42], and Koenderink [31]. Many developments have been proposed, see e.g. [33] for further references on that particular field.

We can see a "scale space" as a family of increasing operators $\{T_t\}_{t\geq 0}$, depending on a scale parameter t. Given an image $u_0(\mathbf{x})$, $(T_t u_0)(\mathbf{x}) = u(t,\mathbf{x})$ is the "image u_0 analyzed (in fact : smoothed) at scale t". For simplicity, $\mathcal{F}$ will be the set of Lipschitz functions on $\mathbb{E} = \mathbb{R}^N$.

We assume that the output at scale t can be computed from the output at a scale $t-h$ for very small h. This is natural, since a coarser analysis of the original picture is likely to be deduced from a finer one without any dependence upon the original picture. By that way the finest picture smoothing is the identity. T_t is obtained by composition of "transition filters", which we denote by $T_{t+h,t}$. For simplicity, we will assume here that $T_{t+h,t}$ will not depend on t, so that one can set $S_h = T_{t+h,t}$. (The general case can be found in [23]). We then say that the scale space $\{T_t\}_{t\geq 0}$ is **pyramidal** if there exists an operator S_h such that for all t one has:

$$T_{t+h} = S_h \circ T_t$$

Note that a much stronger version of the pyramidal structure is the semi-group property already presented in section 1.2.

Since the visual pyramid is assumed to yield more and more global information about the image and its features, it is clear that when the scale increases, no new feature should be created by the scale space : the image at scale t'>t must be simpler than the image at scale t. Furthermore, the transition operator S_h is assumed to act "locally", that is, to look at a small part of the processed image and in a monotone way. In other terms, S_h should be a regular and local increasing operator.

At last, we say that a scale-space $\{T_t\}_{t\geq 0}$ is **causal** if it is pyramidal and if its transition operator S_h is a translation invariant, regular and local increasing operator. To some extent, as increasing operators are the "basic" tools of morphology, causal scale-spaces can be seen as **Morphological Flows**. Operators seen in section 1.2.2 defined examples of causal scale-spaces or "morphological flows".

1.4.1 Causal Scale Space, Increasing Operators and PDEs

We consider a causal scale space $\{T_t\}_{t\geq 0}$ that commutes with addition of constants; i.e., for any constant C, we have $T_t(u + C) = T_t(u) + C$. We denote by F the asymptotic of the transition operator associated to S_h. We know from Eqn. (1.24) that F has the shape: $F(A, p, c)$. The commutation with addition of constants removes the dependence on c, which therefore yields for F a $F(A, p)$ shape.

The next theorems state the equivalence between causal scale-space and viscosity solutions of parabolic PDE. They require some technical assumptions on the shape of the function F that will be given later.

Theorem 1

Let T_t be a causal scale-space. Then for any Lipschitz function u_0: $u(t, .) = T_t(u)(.)$ is the viscosity solution of

$$\frac{\partial u}{\partial t} = F(D^2u, Du) \tag{1.26}$$

with initial condition $u(0, .) = u_0$.

Theorem 2

The operator T_t that associates to a Lipschitz function u_0 the (unique) viscosity solution of the equation (1.26) at scale t is a increasing operator on Lipschitz functions and T_t defines a causal scale-space.

Proofs of these theorem has been given under some regularity conditions on function F. E.g. in [23], Guichard & Morel prove that the preceding theorems hold if F is assumed continuous for all $A, p \neq 0$ and such that there exists two continuous functions $G^+(A,p)$ and $G^-(A,p)$, with $G^+(0,0) = G^-(0,0) = 0$; $\forall A \geq 0,\ G^+(A,0) \geq 0$ and $G^-(-A,0) \leq 0$ and $\forall A, p,\ \ G^-(A,p) \leq F(A,p) \leq G^+(A,p)$. These conditions are in fact not so restrictive since they are satisfied by all equations mentioned in the present chapter.

1.4.2 Geometric and Contrast Invariant Scale Spaces

We shall now list a series of axioms which state some invariance for the scale space. We begin by considering a "contrast invariance" assumption, that the scale space should be independent from the (arbitrary) graylevel scale. We shall say that a scale space is **contrast invariant** if

$$g \circ T_t = T_t \circ g, \tag{1.27}$$

for any nondecreasing and continuous function g from $\mathbb{R}$ into $\mathbb{R}$. The contrast invariance is a particular formalization of the invariance of image analysis with respect to changes of illumination. This invariance has been stated in perception theory by Wertheimer [62], as early as 1923. In Mathematical Morphology, the contrast invariance is commented and used e.g. in Serra [56], or by Maragos et al [41]. Within the scale-space framework, Koenderink [31] insists on that invariance but did not proceed due to incompatibility with some imposed linearity property. We will see, in section 1.5, that in addition to this link with perception, "contrast invariance" generates an interesting link between function evolution and set or curve evolution.

Let R be an isometry of $\mathbb{R}^N$ and denote by Ru the function $Ru(\mathbf{x}) = u(R\mathbf{x})$. We shall say that a scale space T_t is **euclidean invariant** if for every isometry R of $\mathbb{R}^N$ into $\mathbb{R}^N$, $RT_t = T_tR$.

Finally, we state an axiom which implies the invariance of the scale space under any affine projection of a planar shape. Set for any such transform $Af(\mathbf{x}) = f(A\mathbf{x})$. We shall say that a scale space T_t is **affine invariant** if for any linear application A of $\mathbb{R}^N$ with $\det(A) = 1$, we have $AT_t = T_tA$.

If we impose the euclidean and contrast invariance, then T_tu_0 obeys a restricted form of the equation (1.26). A general study in dimension N can be found in [20]. We just recall from [2] the two dimensional case.

(i) Let T_t be a euclidean and contrast invariant causal scale space and u_0 be a Lipschitz function, then $u(t) = T_t(u_0)$ is the viscosity solution of

$$\frac{\partial u}{\partial t} = |Du|\beta(curv(u)), \tag{1.28}$$

where β is a continuous nondecreasing real function.

(ii) If the scale space is, in addition, affine invariant, then the only possible equation is, up to a rescaling,

$$\frac{\partial u}{\partial t} = |Du|(curv(u))^{1/3}. \tag{1.29}$$

where, for any C^2 function f and where $Df \neq 0$, $curv(f) = \kappa(f) = div(\frac{Df}{|Df|})$, is the curvature of the level line at the considered point.

Conversely, as proved in [23], the operator T_t that associates to a function u_0 the (unique) viscosity solution of the preceding equations at scale t is a euclidean and contrast invariant increasing operator on Lipschitz functions and the family T_t defines a euclidean and contrast invariant causal scale-space.

1.4.3 Iterations of Increasing Operators and PDEs

We have seen that the causal scale space framework ends up with some particular parabolic equations. However, this very formal definition of scale space might seem very restrictive to be of any interest. Question occurs on how to get a scale space from any scaled increasing operator ?

The following heuristic answers the question:

- choose a increasing operator S, e.g the mean, the median, the dilation, the erosion, etc...
- localize it: S_h, e.g by using equation (1.23),
- iterate it: Set $(T_n)_t = (S_h)^n$ with $hn = t$.

When $n \to \infty$ if the sequence $(T_n)_t$ converges to some operator T_t, then T_t is a causal scale-space. More precisely, consider u_0 a Lipschitz function and set $u_n(t) = (T_n)_t(u_0)$. If $u_n(t)$ converges when n tends to ∞, then $u(t) = lim_{n\to\infty} u_n(t)$ is the viscosity solution of equation (1.26) with F given by the asymptotic of S_h (equation (1.24)).

The shape of function F will necessary inherit from the invariance property of the increasing operator S. E.g. if S is contrast and euclidean invariant, then F is necessarily of the form $F(D^2u, Du) = |Du|\beta(curv(u))$, for some increasing function β.

Unfortunately convergence has not been proved for general forms of local and increasing operators S_h. Let us cite some basic examples: if S_h is the mean filter on a disk of radius h^2, then T_t will solve the heat equation

$$\frac{\partial u}{\partial t} = \Delta u$$

which confirms a well known result. If S_h is a median filter on a disk of radius h^2, then T_t will solve the mean curvature motion

$$\frac{\partial u}{\partial t} = |Du| curv(u) = |Du|\kappa$$

This last equation will be more deeply considered in the following section.

1.5 Curve Evolution and Morphological Flows

Consider at time $t = 0$ an initial simple, smooth, closed planar curve $\Gamma(0)$ that is propagated along its normal vector field at speed V for $t > 0$. Let this evolving curve (front) $\Gamma(t)$ be represented by its position vector $\vec{C}(p,t) = (x(p,t), y(p,t))$ and be parameterized by $p \in [0, J]$ so that it has its interior on the left in the direction of increasing p and $\vec{C}(0,t) = \vec{C}(J,t)$. The curvature along the curve is

$$\kappa = \kappa(p,t) \triangleq \frac{y_{pp}x_p - y_p x_{pp}}{(x_p^2 + y_p^2)^{3/2}}. \tag{1.30}$$

A general front propagation law (flow) is

$$\frac{\partial \vec{C}(p,t)}{\partial t} = V\vec{N}(p,t), \tag{1.31}$$

with initial condition $\Gamma(0) = \{\vec{C}(p,0) : p \in J\}$, where $\vec{N}(p,t)$ is the instantaneous unit *outward normal* vector at points on the evolving curve and $V = \vec{C}_t \cdot \vec{N}$ is the *normal speed*, with $\vec{C}_t = \partial\vec{C}/\partial t$. This speed may depend on local geometrical information such as the curvature κ, global image properties, or other factors independent of the curve. If $V = 1$ or $V = -1$, then $\Gamma(t)$ is the boundary of the dilation or erosion of the initial curve $\Gamma(0)$ by a disk of radius t.

An important speed model, which has been studied extensively by Osher and Sethian [50, 58] for general evolution of interfaces and by Kimia et al. [29] for shape analysis in computer vision, is

$$V = 1 - \epsilon\kappa, \qquad \epsilon \geq 0. \tag{1.32}$$

As analyzed by Sethian [58], when $V = 1$ the front's curvature will develop singularities, and the front will develop corners (i.e., the curve derivatives will develop shocks—discontinuities) at finite time if the initial curvature is anywhere negative. Two ways to continue the front beyond the corners are as follows: (1) If the front is viewed as a geometric curve, then each point is advanced along the normal by a distance t, and hence a "swallowtail" is formed beyond the corners by allowing the front to pass through itself. 2) If the front is viewed as the boundary separating two regions, an *entropy condition* is imposed to disallow the front to pass through itself. In other words, if the front is a propagating flame, then "once a particle is burnt it stays burnt" [58]. The same idea has also been used to model grassfire propagation leading to the medial axis of a shape [7]. It is equivalent to using Huygens' principle to construct the front as the set of points at distance t from the initial front. This can also be obtained from multiscale dilations of the initial front by disks of radii $t > 0$. Both the swallowtail and the entropy solutions are weak solutions. When $\epsilon > 0$, motion with curvature-dependent speed has a smoothing effect. Further, the limit of the solution for the $V = 1 - \epsilon\kappa$ case as $\epsilon \downarrow 0$ is the entropy solution for the $V = 1$ case [58].

To overcome the topological problem of splitting and merging and numerical problems with the Lagrangian formulation of Eq. (1.31), an Eulerian formulation was proposed by Osher and Sethian [50] in which the original curve $\Gamma(0)$ is first embedded in the surface of an arbitrary 2D Lipschitz continuous function $\phi_0(x, y)$ as its level set (contour line) at zero level. For example, we can select $\phi_0(x, y)$ to be equal to the signed distance function from the boundary of $\Gamma(0)$, positive (negative) in the exterior (interior) of $\Gamma(0)$. Then, the evolving planar curve is embedded as the zero-level set of an evolving space-time function $\phi(x, y, t)$:

$$\begin{aligned} \Gamma(t) &= \{(x, y) : \phi(x, y, t) = 0\} && (1.33) \\ \Gamma(0) &= \{(x, y) : \phi_0(x, y, 0) = \phi(x, y) = 0\}. && (1.34) \end{aligned}$$

Geometrical properties of the evolving curve can be obtained from spatial derivatives of the level function. Thus, at any point on the front the curvature and outward normal of the level curves can be found from ϕ (assume $\phi < 0$ over curve interior):

$$\vec{N} = \frac{\nabla\phi}{\|\nabla\phi\|}, \quad \kappa = \operatorname{div}\left(\frac{\nabla\phi}{\|\nabla\phi\|}\right). \tag{1.35}$$

The curve evolution PDE of Eq. (1.31) induces a PDE generating its level function:

$$\begin{aligned} \partial\phi/\partial t &= -V\|\nabla\phi\|, \\ \phi(x, y, 0) &= \phi_0(x, y). && (1.36) \end{aligned}$$

If $V = 1$, the above function evolution PDE is identical to the flat circular erosion PDE of Eq. (1.12) by equating scale with time. Thus, we can view this specific erosion PDE as a special case of the general function evolution PDE of Eq. (1.36) in which all level curves propagate in a homogeneous medium with unit normal speed. Propagation in a heterogeneous medium with a constant-sign $V = V(x, y)$ leads to the eikonal PDE.

1.5.1 Dilation Flows

In general, if B is an arbitrary compact, convex, symmetric planar set of unit scale and if we dilate the initial curve $\Gamma(0)$ with tB and set the new curve $\Gamma(t)$ equal to the outward boundary of $\Gamma(0) \oplus tB$, then this action can also be generated by the following model [4, 54] of curve evolution

$$\frac{\partial \vec{C}}{\partial t} = \mathrm{sptf}_B(\vec{N})\vec{N} \tag{1.37}$$

Thus, the normal speed V, required to evolve curves by dilating them with B, is simply the support function of B evaluated at the curve's normal. Then, in this case the corresponding PDE (1.36) for evolving the level function becomes identical to the general PDE that generates multiscale flat erosions by B, which is given by (1.6) modulo a $(-)$ sign difference.

1.5.2 Curvature Flows

Another important case of curve evolution is when $V = -\kappa$; then,

$$\frac{\partial \vec{C}}{\partial t} = -\kappa \vec{N} = \frac{\partial^2 \vec{C}}{\partial s^2} \tag{1.38}$$

where s is the arc length. This propagation model is known as *Euclidean geometric heat* (or *shortening*) *flow*, as well as *mean curvature motion*. According to some classic results in differential geometry, smooth simple curves, evolving by means of (1.38), remain smooth and simple while undergoing the fastest possible shrinking of their perimeter [18], [19]. Furthermore, any non-convex curve converges first to a convex curve and from there it shrinks to a round point.

If the function $\phi(x, y, t)$ embeds a curve evolving by means of (1.38), as its level curve at a constant level, then it satisfies the evolution PDE

$$\partial\phi/\partial t = \mathrm{div}(\nabla\phi/||\nabla\phi||)||\nabla\phi|| = \kappa||\nabla\phi||$$

This smooths all level curves by propagation under their mean curvature. It has many interesting properties and has been extensively studied by many groups of researchers, including Osher & Sethian [50], Evans & Spruck [17], Chen, Giga & Goto [12] and Alvarez et al. [2].

Solutions of the Euclidean geometric heat flow (1.38) are invariant with respect to the group of Euclidean transformations (rotations and translations). Extending this invariance to affine transformations creates the *affine geometric heat flow* introduced by Sapiro and Tannenbaum [55]

$$\frac{\partial \vec{C}}{\partial t} = -\kappa^{1/3}\vec{N} = \frac{\partial^2 \vec{C}}{\partial \alpha^2} \tag{1.39}$$

where α is the affine arc length, i.e., a re-parameterization of the curve such that $\det[\vec{C}_\alpha \;\, \vec{C}_{\alpha\alpha}] = x_\alpha y_{\alpha\alpha} - x_{\alpha\alpha} y_\alpha = 1$. Any smooth simple non-convex curve evolving by the affine flow (1.39) converges to a convex one and from there to an elliptical point [55]. This PDE was also independently developed by Alvarez et al. [2] in the context of the affine morphological scale-space, already seen in section 1.4.2.

1.5.3 Morphological Representations of Curvature Flows

Matheron's famous representation theorem [43] states that any set operator Ψ on $\mathcal{P}(\mathbb{R}^d)$ that is translation-invariant (TI) and increasing can be represented as the union of erosions by all sets of its kernel $\mathrm{Ker}(\Psi) = \{X : \vec{0} \in \Psi(X)\}$ as well as an intersection of dilations by all sets of the kernel of the dual operator:

$$\Psi \text{ is TI and increasing} \implies \Psi(X) = \bigcup_{A \in \mathrm{Ker}(\Psi)} X \ominus A, \quad X \subseteq \mathbb{R}^d.$$

This representation theory was extended by Maragos [35, 36] to both function and set operators by using a basis for the kernel. As we have seen in section 1.3.1, according to the basis representation theory, every *TI, increasing*, and *upper-semicontinuous* (u.s.c.) operator can be represented as a supremum of morphological erosions by its basis functions. Specifically, let ψ be a signal operator acting on the set of extended-real-valued functions defined on $\mathbb{E} = \mathbb{R}^d$ or $\mathbb{Z}^d$. If $\mathrm{Ker}(\psi) = \{f : \psi(f)(\vec{0}) \geq 0\}$ defines the *kernel* of ψ, then its *basis* $\mathrm{Bas}(\psi)$ is defined as the collection of the minimal (w.r.t. $\leq$) kernel functions. Then [36]:

$$\psi \text{ is TI, increasing, and u.s.c.} \implies \psi(f) = \bigvee_{g \in \mathrm{Bas}(\psi)} f \ominus g$$

Dually, ψ can be represented as the infimum of dilations by functions in the basis of its dual operator $\psi^*(f) = -\psi(-f)$.

If the above function operator ψ is also flat (i.e., binary inputs yield binary outputs), with Ψ being its corresponding set operator, and commutes with thresholding, i.e.,

$$X_\lambda[\psi(f)] = \Psi[X_\lambda(f)], \quad \lambda \in \mathbb{R} \tag{1.40}$$

where $X_\lambda(f) = \{x \in \mathbb{R}^d : f(x) \geq \lambda\}$ are the *upper level sets* of f, then ψ is a supremum of flat erosions by the basis sets of its corresponding set operator Ψ [36]:

$$\psi(f) = \bigvee_{S \in \mathrm{Bas}(\Psi)} f \ominus S$$

where the basis $\mathrm{Bas}(\Psi)$ of the set operator Ψ is defined as the collection of the minimal elements (w.r.t. $\subseteq$) of its kernel $\mathrm{Ker}(\Psi)$.

Equation (1.40) implies that [57, p. 188] the operator ψ is 'contrast-invariant' or 'morphologically-invariant,' which means that [56, 1, 22]

$$\psi(g(f)) = g(\psi(f))$$

where $g : \mathbb{R} \to \mathbb{R}$ is any monotone bijective function, and $g(f)$ is the image of f under g. Such a function g is called an 'anamorphosis' in [56, 57], or a 'contrast-change' in [1, 22].

The above morphological basis representations have been applied to various classes of operators, including morphological, median, stack, and linear filters [35, 36, 41]. Moreover, one can define TI, increasing and contrast-invariant filters as supremum (or infimum) of flat erosions (or dilations) by sets belonging to some arbitrary basis $\mathbb{B}$. Catté, Dibos & Koepfler [11] selected as a basis the scaled version of a unit-scale isotropic basis (the set of all symmetric line segments of length 2)

$$\mathbb{B} \triangleq \{\{(x, y) : y = x\tan(\theta), |x| \leq |\cos(\theta)|\} : \theta \in [0, \pi)\} \tag{1.41}$$

and defined the following three types of multiscale flat operators $\mathcal{I}_t, \mathcal{S}_t, \mathcal{T}_t$:

$$\mathcal{I}_t(f) = \bigvee_{S \in \mathbb{B}} f \ominus \sqrt{2t}S \iff \partial\vec{C}/\partial t = -\max(\kappa, 0)\vec{N} \tag{1.42}$$

$$\mathcal{S}_t(f) = \bigwedge_{S \in \mathbb{B}} f \oplus \sqrt{2t}S \iff \partial\vec{C}/\partial t = \min(\kappa, 0)\vec{N} \tag{1.43}$$

$$\mathcal{T}_t(f) = [\mathcal{I}_{2t}(f) + \mathcal{S}_{2t}(f)]/2 \iff \partial\vec{C}/\partial t = -\kappa\vec{N} \tag{1.44}$$

If these operators operate on a level function embedding a curve $\vec{C}$ as one of its level lines, then this curve evolves according to the above following three flows [11]. Hence, the above multiscale operators, which are sup-of-erosions and inf-of-dilations by linear segments in all directions, are actually curvature flows. A generalization of this result was obtained, within the framework described in section 1.4, in Guichard and Morel [22], by assuming that $\mathbb{B}$ is any bounded and isotropic collection of planar sets. Furthermore, in slightly different settings it has been shown that, by iterating n times a median filter, based on a window of scale h, we asymptotically converge (when $h \to 0$, $n \to \infty$, with $nh = t$) to the curvature flow. The mathematical proof was given in [16], [6], following a conjecture of [45].

The above morphological representations deal with Euclidean curvature flow. Furthermore, by defining a unit-scale morphological basis $\mathbb{B}$ as a collection of convex symmetric sets invariant under the special linear group, it has been shown in [22] and in [20] that n iterations of morphological flat operators at scale h, which are sup-of-erosions, inf-of-dilations, or their alternate compositions, converge (when $h \to 0$, $n \to \infty$, with $nh = t$) to the affine curvature flow. An efficient implementation of the iterated affine invariant curve evolution has been proposed in [48]. It yields a fast implementation of the curve affine scale space and has proved its effectiveness in shape recognition [34]. An example of shape smoothing using this affine scale-space is shown in Fig. 1.1.

1.6 Conclusion

In this chapter we have presented some basic results from the theory of nonlinear geometric PDEs that can generate multiscale morphological operators. Further, we have outlined the relationships of these results with G. Matheron's development of size distributions and kernel representation theory.

Interpreting and modelling the basic morphological operators via PDEs opens up several new promising directions along which mathematical morphology can both assist and be assisted by other PDE-based theories and methodologies of image analysis and computer vision. Examples include scale-space analyses, variational methods of vision, level sets implementations of 2D/3D geometric flows, and their applications to major research problems such as image segmentation, object detection & tracking, and stereopsis.

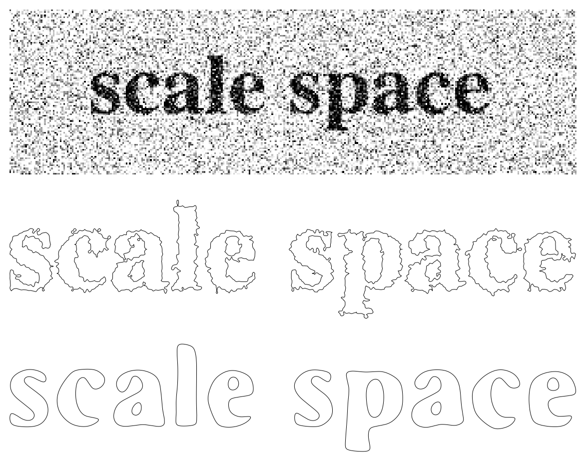

Figure 1.1: Smoothing curves with the Affine Scale Space. Top: a text image corrupted by noise. Middle: thresholding the image reveals characters as irregular level lines. Bottom: the same level lines, smoothed with the affine scale space. The smoothing process produces curves almost independent of the noise, which is a requirement for robust pattern recognition. Algorithm used follows the affine erosion introduced in [48]. -Experiment courtesy of Lionel Moisan-

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