On Amplitude and Frequency Demodulation Using Energy Operators

Petros Maragos, *Senior Member, IEEE*, James F. Kaiser, *Fellow, IEEE*, and Thomas F. Quatieri, *Senior Member, IEEE*

***Abstract*—Amplitude-modulation (AM) and frequency-modulation (FM) systems are widely applicable to the modeling and transmission of information in signals. In this paper it is shown that the nonlinear energy-tracking signal operator $\Psi(x) = (\dot{x})^2 - x\ddot{x}$ and its discrete-time counterpart can estimate the AM and FM modulating signals. Specifically, Ψ can approximately estimate the amplitude envelope of AM signals and the instantaneous frequency of FM signals. Bounds are derived for the approximation errors, which are negligible under general realistic conditions. These results, coupled with the simplicity of Ψ, establish the usefulness of the energy operator for AM and FM signal demodulation. These ideas are then extended to a more general class of signals that are sine waves with a time-varying amplitude and frequency and thus contain both an AM and an FM component; for such signals it is shown that Ψ can approximately track the product of their amplitude envelope and their instantaneous frequency. The theoretical analysis is done for both continuous- and discrete-time signals.**

I. Introduction

In his extensive work on nonlinear speech modeling, Teager [14]–[16] noted the dominance of modulation as a process in the production of speech. He also noted the importance of analyzing speech signals from the point of view of the energy required to generate them. One of the beautifully simple algorithms for signal analysis he devised and used extensively was the following nonlinear energy-tracking operator Ψ given either in its discrete form Ψ_d when operating on discrete-time signals $x(n)$:

$$\Psi_d[x(n)] \triangleq x^2(n) - x(n+1)x(n-1) \tag{1}$$

Manuscript received May 6, 1991; revised May 29, 1992. The work of P. Maragos was supported by the National Science Foundation under Grant MIP-91-20624, by a National Science Foundation Presidential Young Investigator Award under Grant MIP-86-58150 with matching funds from Bellcore, DEC, and Xerox, and in part by the Army Research Office Grant DAAL03-86-K-0171 to the Brown-Harvard-MIT Center for Intelligent Control Systems. The work of T. F. Quatieri was supported in part by the Department of the Air Force and in part by the Naval Submarine Medical Research Laboratory.

P. Maragos is with the Division of Applied Sciences, Harvard University, Cambridge, MA 02138.

J. F. Kaiser was with Bellcore, Morristown, NJ 07962. He is now with the Department of Electrical and Computer Engineering, Rutgers University, Piscataway, NJ 08855.

T. F. Quatieri is with the M.I.T. Lincoln Laboratory, Lexington, MA 02173.

IEEE Log Number 9206857.

or in its continuous form Ψ_c when operating on continuous-time signals $x(t)$:

$$\Psi_c[x(t)] \triangleq \left(\frac{dx(t)}{dt}\right)^2 - x(t)\left(\frac{d^2x(t)}{dt^2}\right) = [\dot{x}(t)]^2 - x(t)\ddot{x}(t) \tag{2}$$

where $\dot{x} = dx/dt$. Both of these operators were first introduced systematically by Kaiser [2]–[4].

Ψ_c can track the energy of a linear oscillator; thus it can be viewed as an energy operator. The reasoning [2], [3] proceeds as follows: Consider an undriven linear undamped oscillator consisting of a mass m and a spring of constant k. Its displacement $x(t)$ is governed by the motion equation $m\ddot{x} + kx = 0$, for which the general solution is a cosine $x(t) = A\cos(\omega_0 t + \theta)$ with $\omega_0 = \sqrt{k/m}$. The instantaneous energy E_0 of this undamped oscillator is constant and equal to the sum of its kinetic and potential energy; i.e.,

$$E_0 = \frac{m}{2}(\dot{x})^2 + \frac{k}{2}x^2 = \text{constant} = \frac{m}{2}(A\omega_0)^2. \tag{3}$$

Thus the energy of the linear oscillator is proportional both to the amplitude squared and the frequency squared of the oscillation, a fact noted in [9]. Further,

$$\Psi_c[A\cos(\omega_0 t + \theta)] = A^2\omega_0^2 = \frac{E_0}{(m/2)}. \tag{4}$$

Namely, when Ψ_c is applied to the oscillation signal, the output of Ψ_c is equal to the energy (per half unit mass) of the source producing the oscillation.

Kaiser [3], [4] found the following important properties of Ψ_c: for any constants A, r, and ω_0 and for any signals x and y

$$\Psi_c[Ae^{rt}\cos(\omega_0 t + \theta)] = A^2e^{2rt}\omega_0^2 \tag{5}$$

$$\Psi_c[x(t)y(t)] = x^2(t)\Psi_c[y(t)] + y^2(t)\Psi_c[x(t)] \tag{6}$$

The discrete operator also has similar properties [2], [3]:

$$\Psi_d[Ar^n\cos(\Omega_0 n + \theta)] = A^2r^{2n}\sin^2(\Omega_0) \tag{7}$$

$$\Psi_d[x(n)y(n)] = x^2(n)\Psi_d[y(n)] + y^2(n)\Psi_d[x(n)] - \Psi_d[x(n)]\Psi_d[y(n)]. \tag{8}$$

In this paper we apply the operators Ψ_c and Ψ_d to the broad class of amplitude-modulated (AM) and frequency-modulated (FM) signals. (For the fundamentals of AM and FM systems see any book on communications, e.g., [13].) Since we analyze both continuous- and discrete-time signals, first the relationship between Ψ_c and Ψ_d is investigated in Section II. Then, in Section III, we show how these energy operators can approximately track the envelope of AM signals. In Section IV they are found to be able to approximately estimate the instantaneous frequency of FM signals. For all these results we also provide the magnitude of the approximation errors involved. Section V deals with the generalized case of AM–FM signals, i.e., FM signals with a time-varying amplitude envelope, or equivalently AM signals with a time-varying instantaneous frequency. For such signals we show that the energy operators can approximately track the product of their amplitude envelope and their instantaneous frequency. In Section VI it is shown that various general classes of AM–FM signals yield a nonnegative signal in the output of the energy operators; such a guarantee for the nonnegativity of the energy operators is needed for using them in AM and FM demodulation. In Section VII we conclude and outline some extensions and applications of our work.

Given the simplicity of the energy operators and the broad applicability of AM and FM models in signal processing and communications systems, the results derived in this paper are very useful. For example, some results in this paper have been used in the ongoing work of Maragos, Quatieri, and Kaiser [5], [6] on speech modeling using an AM–FM model where, inspired by Teager's work, several amplitude/frequency modulation phenomena in time-varying speech resonances are being investigated. In addition, our results in this work have been used by Quatieri *et al.* [12] in detecting transient signal signatures corrupted by AM–FM noise. In this paper, however, we do not assume anything about the specific nature or source of the AM or FM signals.

Throughout the paper we deal with real-valued signals. Finally, although we treat both discrete- and continuous-time signals and operators, our most general results and intuition are derived for and from the continuous-time case.

II. Discretizing the Continuous-Time Operator

By using certain combinations of discretized derivatives we can obtain from Ψ_c an expression closely related to Ψ_d and thus link the two operators. We examined three cases.

A. Two-Sample Backward Difference

We replace t with nT_s (T_s is the sampling period), $x(t)$ with $x(nT_s)$ or simply $x(n)$, $\dot{x}(t)$ with $y(n) = [x(n) - x(n-1)]/T_s$ and $\ddot{x}(t)$ with $[y(n) - y(n-1)]/T_s$. Then

$$\dot{x}(t) \mapsto [x(n - x(n-1)]/T_s$$
$$\ddot{x}(t) \mapsto [x(n) - 2x(n-1) + x(n-2)]/T_s^2$$
$$\Psi_c[x(t)] \mapsto \Psi_d[x(n-1)]/T_s^2.$$

where $\mapsto$ denotes the mapping from continuous to discrete. Thus from Ψ_c we obtained Ψ_d shifted by one sample to the right and scaled by T_s^{-2}.

B. Two-Sample Forward Difference

$$\dot{x}(t) \mapsto [x(n+1) - x(n)]/T_s$$
$$\ddot{x}(t) \mapsto [x(n+2) - 2x(n+1) + x(n)]/T_s^2$$
$$\Psi_c[x(t)] \mapsto \Psi_d[x(n+1)]/T_s^2.$$

C. Three-Sample Symmetric Difference

$$\dot{x}(t) \mapsto [(x(n+1) - x(n)) + (x(n) - x(n-1))]/2T_s = [x(n+1) - x(n-1)]/2T_s$$
$$\ddot{x}(t) \mapsto [x(n+2) - 2x(n) + x(n-2)]/4T_s^2$$
$$\Psi_c[x(t)] \mapsto (\Psi_d[x(n+1)] + 2\Psi_d[x(n)] + \Psi_d[x(n-1)])/4T_s^2.$$

Thus, if we ignore the one-sample shift and the scaling by T_s^{-2}, both asymmetric two-sample differences succeed in transforming $\Psi_c[x(t)]$ into $\Psi_d[x(n)]$. However, the three-sample symmetric difference gives a more complicated expression; i.e., it results into a three-sample weighted moving average of $\Psi_d[x(n)]$.

Henceforth, we shall drop the subscripts c and d from Ψ since it will be clear from the context whether we refer to continuous or discrete time.

III. Amplitude Modulation (AM)

A. Continuous-Time AM

Let the general real-valued AM signal be

$$x(t) = a(t)\cos(\omega_c t + \theta)$$

where $\omega_c > 0$ is the constant carrier angular frequency, and θ is a constant phase offset. The amplitude signal $a(t)$ is either proportional to or contains the AM "modulating signal" (also called "information signal") and usually varies more slowly than the carrier signal $\cos(\omega_c t)$. The AM envelope is defined as the signal $|a(t)|$. Now consider Ψ applied to the AM signal. By properties (6) and (5) we obtain

$$\Psi[a(t)\cos(\omega_c t + \theta)] = \underbrace{a^2(t)\omega_c^2}_{D(t)} + \underbrace{\cos^2(\omega_c t + \theta)\Psi[a(t)]}_{E(t)}. \quad (9)$$

Since we are interested in using Ψ to estimate the envelope contained in the term D, we next find sufficient conditions under which the desired term D dominates over the term E; then we can view E as an approximation error. To quantify this error we compare an amplitude scale characteristic of the order of magnitude of the dominant signal term with a similar amplitude scale of the error signal term. For example, if we want to restrict the maximum absolute value of the error E to be much smaller than the value of the desired term D at each time instant, then we demand that $E_{\max} << D_{\min}$, where for an arbitrary signal $x(t)$ we denote

$$x_{\max} \triangleq \sup_t \{|x(t)|\}, \qquad x_{\min} \triangleq \inf_t \{|x(t)|\}.$$

Since for all t

$$|E(t)| \le |\Psi(a(t))| \tag{10}$$

it follows that

$$\Psi[a(t) \cos(\omega_c t + \theta)] \approx [\omega_c a(t)]^2, \quad \text{if } [\Psi(a)]_{\max} << (\omega_c a_{\min})^2 \tag{11}$$

where $\approx$ means "approximately equal." The above approximating condition is meaningful only if $D_{\min} > 0$, i.e., if $a_{\min} > 0$. (Note that in general $D(t) \ge 0$ for all t).

In several applications, for example in cases where $a_{\min} = 0$ or in cases where we need to quantify the average error, we may want to compare the mean absolute value of the error E with that of D. By "mean" we refer to a "time average" defined as follows. For an aperiodic signal $x(t)$, which is absolutely integrable, we define

$$x_{\text{ave}} \triangleq \int_{-\infty}^{\infty} x(t)\, dt.$$

Otherwise, if x is periodic with period T, or if we analyze an arbitrary signal x only over the finite time interval $[0, T]$, then we define

$$x_{\text{ave}} \triangleq \frac{1}{T} \int_0^T x(t)\, dt.$$

In each case, the root-mean square (rms) value of x is defined by

$$x_{\text{rms}} \triangleq \sqrt{(x^2)_{\text{ave}}}.$$

Now returning to (9), we will also consider approximations

$$\Psi[a(t) \cos(\omega_c t + \theta)] \approx_{\text{ave}} [\omega_c a(t)]^2, \quad \text{if } |\Psi(a)|_{\text{ave}} << (\omega_c a_{\text{rms}})^2 \tag{12}$$

where $\approx_{\text{ave}}$ denotes an approximation with a very small mean absolute error.

To quantify the relative approximation error of (11) and (12) we define two types of a signal-to-error ratio (SER) in the output of Ψ, where by "signal" we mean the desired energy term D. The instantaneous SER

$$\text{ISER}(t) \triangleq \frac{D(t)}{|E(t)|}$$

is defined at each t except for the isolated time instants when both D and E are zero. The average SER

$$\text{ASER} \triangleq \frac{D_{\text{ave}}}{|E|_{\text{ave}}}$$

is defined over the whole time interval that corresponds to the averaging operation. Note that these SER's have the following general lower bounds:

$$\text{ISER}(t) \ge \frac{D_{\min}}{E_{\max}} \ge \frac{(\omega_c a_{\min})^2}{\Psi(a)_{\max}} \tag{13}$$

$$\text{ASER} \ge \frac{(\omega_c a_{\text{rms}})^2}{|\Psi(a)|_{\text{ave}}}. \tag{14}$$

The approximate result in (11) or (12) will be meaningful and useful only if the instantaneous or the average SER is $>> 1$. Before we show that this is true under very broad assumptions about the amplitude signal, we first provide some required results concerning the maximum absolute value of signals and their derivatives. Throughout the paper, we shall use the notation $X(\omega)$ for the Fourier transform of an aperiodic signal $x(t)$, where ω denotes angular frequency, and the Fourier series representation $x(t) = \Sigma_k \alpha_k e^{jk\omega_0 t}$ if $x(t)$ is periodic with fundamental period $2\pi/\omega_0$ and α_k being its kth Fourier series coefficient. In addition, we shall use the spectral absolute moments[1] of real-valued signals $x(t)$ defined as

$$\mu_{x,n} \triangleq \begin{cases} \dfrac{1}{\pi} \displaystyle\int_0^{\infty} \omega^n |X(\omega)|\, d\omega, & \text{if } x(t) \text{ is aperiodic} \\ \omega_0^n \displaystyle\sum_{k=-\infty}^{\infty} |k|^n |\alpha_k|, & \text{if } x(t) = \displaystyle\sum_k \alpha_k e^{jk\omega_0 t} \end{cases}$$

for $n = 0, 1, 2, \cdots$. The zeroth moment (which is frequently used in this paper) is denoted by

$$\mu_x = \mu_{x,0}.$$

For aperiodic signals x it is known [10, p. 34] that for all t

$$\left| \frac{d^n x(t)}{dt^n} \right| \le \mu_{x,n}. \tag{15}$$

This also holds for periodic $x(t) = \Sigma_k \alpha_k e^{jk\omega_0 t}$ since the kth Fourier coefficient of $d^n x/dt^n$ is $(jk\omega_0)^n \alpha_k$. As a special case of (15) for $n = 0$, we have

$$|x(t)| \le x_{\max} \le \mu_x. \tag{16}$$

Note that $x_{\max} = \mu_x$ if x has a linear Fourier phase; i.e.,

$$X(\omega) = \pm|X(\omega)| e^{-j\omega t_0} \Rightarrow x_{\max} = \mu_x \tag{17}$$

because, as shown in [10, p. 65],

$$x_{\max} = |x(t_0)| = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|\, d\omega.$$

Similarly, for periodic $x(t)$ we have

$$x(t) = \pm \sum_{k=0}^{\infty} |A_k| \cos[k\omega_0(t - t_0)] \Rightarrow x_{\max} = \mu_x \tag{18}$$

[1] In this paper whenever we use one of the moments $\mu_{x,n}$ it will be assumed that it is finite.

where t_0 and A_k are arbitrary real constants. As a special case, for any cosine $x(t) = A \cos(\omega_0 t)$ we have $x_{\max} = \mu_x = |A|$.

Finally, for band limited signals x with highest frequency ω_x it follows[2] from (15) that

$$X(\omega) = 0 \quad \forall |\omega| > \omega_x \Rightarrow \left| \frac{d^n x(t)}{dt^n} \right| \leq (\omega_x)^n \mu_x \quad (19)$$

for $n \geq 1$.

The previous results allow us to find next the maximum absolute or average value of the signal in the output of the energy operator.

Proposition 1: Let $x(t)$ be a continuous-time real-valued signal with finite spectral moments $\mu_{x,n}$ for $n = 0, 1, 2$. Then (a)

$$|\Psi(x(t))| \leq (\mu_{x,1})^2 + \mu_{x,0}\mu_{x,2} \quad (20)$$

for all t. Further, if x is aperiodic,

$$\int_{-\infty}^{\infty} \Psi[x(t)]\, dt = \frac{2}{\pi} \int_0^{\infty} \omega^2 |X(\omega)|^2\, d\omega. \quad (21)$$

If $x(t) = \Sigma_k \alpha_k e^{jk\omega_0 t}$ is periodic with period $T = 2\pi/\omega_0$,

$$\frac{1}{T} \int_0^T \Psi[x(t)]\, dt = 2\omega_0^2 \sum_{k=-\infty}^{\infty} k^2 |\alpha_k|^2. \quad (22)$$

(b) If x is also band limited with highest frequency ω_x,

$$|\Psi(x(t))| \leq 2(\omega_x \mu_x)^2 \quad (23)$$

$$\Psi(x)_{\text{ave}} \leq 2(\omega_x x_{\text{rms}})^2. \quad (24)$$

Proof: (a) Equation (20) results from (15) and

$$|\Psi(x(t))| \leq |\dot{x}^2(t)| + |x(t)\ddot{x}(t)|. \quad (25)$$

For (21) note that, by Parseval's theorem for aperiodic signals x,

$$\int_{-\infty}^{\infty} [(\dot{x}(t))^2 - x(t)\ddot{x}(t)]\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\omega^2 |X(\omega)|^2\, d\omega. \quad (26)$$

Similarly, (22) results from Parseval's theorem for periodic signals.

(b) By using (19) and (25) we obtain (23). To prove (24) for aperiodic signals x, note that, by (21),

$$\Psi(x)_{\text{ave}} = \frac{2}{\pi} \int_0^{\omega_x} \omega^2 |X(\omega)|^2\, d\omega \leq \frac{2\omega_x^2}{\pi} \int_0^{\omega_x} |X(\omega)|^2\, d\omega = 2(\omega_x x_{\text{rms}})^2.$$

Similarly, for the case of periodic signals $x(t) = \Sigma_k \alpha_k e^{jk\omega_0 t}$, where $\alpha_k = 0$ if $|k| > \omega_x/\omega_0$ and hence

$$(x_{\text{rms}})^2 = \sum_{k=-\omega_x/\omega_0}^{\omega_x/\omega_0} |\alpha_k|^2.$$

This completes the proof. Q.E.D.

[2]The bounds for the derivatives of band-limited signals are also related to Berstein's theorem [1].

Proposition 1 paved the way to find the following bounds for the approximation error and the corresponding SER's involved in envelope tracking via the energy operator.

Theorem 1: Consider a continuous-time AM signal $x(t) = a(t) \cos(\omega_c t + \theta)$, where $a(t)$ is real valued, band limited with highest frequency ω_a, and has finite spectral moments $\mu_{a,n}$ for $n = 0, 1, 2$. Let $E(t) = \Psi[x(t)] - \omega_c^2 a^2(t)$ be the approximation error. Then: (a)

$$|E(t)| \leq 2(\omega_a \mu_a)^2. \quad (27)$$

If $\Psi[a(t)] \geq 0$ for all t, the average SER in the output of $\Psi(x)$ is bounded as

$$\text{ASER} \geq \frac{1}{2} \left(\frac{\omega_c}{\omega_a} \right)^2. \quad (28)$$

(b) If $a(t) = \cos(\omega_a t)$, then $|E(t)| \leq \omega_a^2$.

Proof: (a) (27) immediately follows from (10) and (23). Further, (28) results from (14) and (24). For (b) note that

$$\Psi[\cos(\omega_a t) \cos(\omega_c t + \theta)] = \omega_c^2 \cos^2(\omega_a t) + \omega_a^2 \cos^2(\omega_c t + \theta) \quad (29)$$

Hence, $|E(t)| \leq \omega_a^2$. Q.E.D.

Theorem 1 implies that, if we assume a band-limited signal $a(t)$ with bandwidth $\omega_a \ll \omega_c$, then $|E|_{\text{ave}} \ll D_{\text{ave}}$ and the energy operator can track the AM envelope squared (within a multiplicative constant) with a very small mean absolute error. Then, applying a square root operation after Ψ yields a signal proportional to the actual envelope $|a(t)|$. Note that in taking the square root we henceforth assume that the output of Ψ is a nonnegative signal. This assumption was also used for deriving the lower bound (28). As discussed later in Section VI, we have found large classes of useful signals and sufficient conditions for general signals x for which $\Psi(x)$ is nonnegative.

An important special case for $a(t)$ that we examine below is $a(t) = 1 + \kappa b(t)$, where $b(t)$ is the modulating (or information) signal, κ is the AM index, and we henceforth assume that

$$-1 \leq b(t) \leq 1 \quad \forall t, \qquad 0 < \kappa \leq 1.$$

Such AM signals are called AM with carrier (AM/WC). Actually, all AM signals that have amplitude $a(t) \geq 0$ for all t can be viewed as equivalent to an AM/WC signal with amplitude

$$A[1 + \kappa b(t)] = a(t)$$

where

$$A = \frac{a_{\max} + a_{\min}}{2}, \qquad \kappa = \frac{a_{\max} - a_{\min}}{a_{\max} + a_{\min}},$$

$$b(t) = \frac{a(t) - A}{A\kappa}.$$

In contrast, AM signals without a carrier are often called AM with suppressed carrier (AM/SC); in this case $a(t)$ assumes both positive and negative values. Using Ψ to estimate the envelope of AM/WC signals has three advantages over using it on AM/SC signals: i) The nonnegative amplitude $a(t)$ is identical with the envelope $|a(t)|$, and hence the modulating signal $b(t)$ can be directly extracted from the estimated envelope. Thus Ψ can be used for demodulation, i.e., recovery of the modulating signal $b(t)$, in the case of AM/WC signals. ii) If $\kappa < 1$, we have $a_{\min} > 0$ and can derive nonzero lower bounds for the instantaneous SER in the output of Ψ, which is a useful measure of the relative approximation error because of the instantaneously adapting nature of Ψ. iii) For AM/WC signals we will be able to find tighter error bounds than the general bounds derived in Theorem 1.

Theorem 2: Consider a continuous-time AM/WC signal $x(t) = a(t) \cos(\omega_c t + \theta)$, where $a(t) = 1 + \kappa b(t)$, and $b(t)$ is real valued, band limited with highest frequency ω_a, and has finite spectral moments $\mu_{b,n}$ for $n = 0, 1, 2$. Then: (a) the approximation error $E(t) = \Psi[x(t)] - \omega_c^2 a^2(t)$ has the following upper bound:

$$|E(t)| \leq \kappa \omega_a^2 (\mu_b + 2\kappa \mu_b^2). \tag{30}$$

The instantaneous SER in the output of $\Psi(x)$ has the following lower bound:

$$\text{ISER}(t) \geq \left(\frac{\omega_c}{\omega_a}\right)^2 \left[\frac{(1-\kappa)^2}{\kappa(\mu_b + 2\kappa\mu_b^2)}\right]. \tag{31}$$

(b) Over any finite time interval $[0, T]$ on which $\int_0^T b(t)\, dt = 0$, the average SER has the following lower bound:

$$\text{ASER} \geq \left(\frac{\omega_c}{\omega_a}\right)^2 \left[\frac{1 + (\kappa b_{\text{rms}})^2}{\kappa(\mu_b + 2\kappa\mu_b^2)}\right]. \tag{32}$$

If $b(t)$ is also periodic with period T and $\Psi(b) \geq 0$, then

$$\text{ASER} \geq \left(\frac{\omega_c}{\omega_a}\right)^2 \left[\frac{1 + (\kappa b_{\text{rms}})^2}{\kappa b_{\text{rms}}(1 + 2\kappa b_{\text{rms}})}\right] \tag{33}$$

$$\geq \left(\frac{\omega_c}{\omega_a}\right)^2 \left(\frac{1}{\kappa + 2\kappa^2}\right). \tag{34}$$

(c) If $b(t) = \cos(\omega_a t)$,

$$|E(t)| \leq \kappa(1+\kappa)\omega_a^2 \tag{35}$$

$$\text{ISER}(t) \geq \left(\frac{\omega_c}{\omega_a}\right)^2 \left[\frac{(1-\kappa)^2}{\kappa(1+\kappa)}\right]. \tag{36}$$

Proof: (a) From the general property

$$\Psi[1 + \kappa b(t)] = \kappa^2 \Psi[b(t)] - \kappa \ddot{b}(t) \tag{37}$$

and (10) we obtain

$$|E(t)| \leq \kappa^2 |\Psi(b(t))| + \kappa |\ddot{b}(t)| \tag{38}$$

from which (30) results due to Proposition 1. Since $a_{\min} = 1 - k$, (31) follows from (13) and (30).

(b) By (30),

$$|E|_{\text{ave}} = \frac{1}{T}\int_0^T |E(t)|\, dt \leq E_{\max} \leq \kappa\omega_a^2(\mu_b + 2\kappa\mu_b^2). \tag{39}$$

Now since $a^2 = 1 + (\kappa b)^2 + 2\kappa b$ and $b_{\text{ave}} = 0$, we have

$$D_{\text{ave}} = \frac{\omega_c^2}{T}\int_0^T [1 + \kappa b(t)]^2\, dt = \omega_c^2[1 + (\kappa b_{\text{rms}})^2]. \tag{40}$$

The two previous results prove (32).

If b is also periodic, then $\Psi(b)_{\text{ave}} \leq 2(\omega_a b_{\text{rms}})^2$ and (by using Cauchy-Schwarz's inequality and Parseval's theorem)

$$|\ddot{b}|_{\text{ave}} \leq (\ddot{b})_{\text{rms}} \leq \omega_a^2 b_{\text{rms}}. \tag{41}$$

Hence, if $\Psi(b) \geq 0$, by (38)

$$|E|_{\text{ave}} \leq \kappa^2\Psi(b)_{\text{ave}} + \kappa|\ddot{b}|_{\text{ave}} \leq \kappa\omega_a^2 b_{\text{rms}}(1 + 2\kappa b_{\text{rms}}). \tag{42}$$

This and (40) prove (33), from which (34) follows since $b_{\text{rms}} \leq 1$.

(c) If $b(t) = \cos(\omega_a t)$, then (37) implies that

$$\Psi[1 + \kappa\cos(\omega_a t)] = (\kappa\omega_a)^2 + \kappa\omega_a^2\cos(\omega_a t). \tag{43}$$

Hence

$$|\Psi(1 + \kappa\cos(\omega_a t))| \leq \kappa(1+\kappa)\omega_a^2 \tag{44}$$

This and the results from (a) and (b) complete the proof of (c). Q.E.D.

Thus, the envelope estimation by the energy operator Ψ in AM/WC signals can yield a smaller upper bound for the error (and equivalently a larger lower bound for the SER) than when Ψ is applied to AM/SC signals. For example, assuming at $\mu_b \approx 1$ or that $b(t)$ is periodic, the lower bound for the average SER is larger for AM/WC signals if $\kappa \leq 0.78$. In addition, $\text{ISER}(t) \geq (\omega_c^2/\kappa\omega_a^2)$ whenever $\kappa << 1$. Therefore, assuming that $\mu_b \approx 1$ (which is true with equality if b has linear phase) and $\omega_a < \omega_c$, the instantaneous SER will be $>> 1$ if $\omega_a << \omega_c$ (a standard assumption in AM applications) and $\kappa << 1$ (a low percent of AM), or more generally if $[\omega_c(1-\kappa)]^2 >> \omega_a^2(\kappa + 2\kappa^2)$; under such conditions

$$\sqrt{\Psi[(1 + \kappa b(t))\cos(\omega_c t + \theta)]} \approx \omega_c[1 + \kappa b(t)]$$

which demonstrates the envelope tracking abilities of $\sqrt{\Psi}$.

Note that if we apply Ψ to AM/WC signals with $\kappa < 1$ and the instantaneous SER in the output of Ψ is $>> 1$ for all t, then we can also quantify the relative error in the output of the $\sqrt{\Psi}$ operator. Specifically, in the output of $\sqrt{\Psi}$ the desired signal term is $\sqrt{D}$, whereas the approximation error term is $\sqrt{D + E} - \sqrt{D}$. Hence, the instantaneous SER in the output of $\sqrt{\Psi}$ is

$$\text{ISER}_{\sqrt{\Psi}}(t) \triangleq \frac{\sqrt{D(t)}}{|\sqrt{D(t)+E(t)} - \sqrt{D(t)}|}.$$

In general, it is difficult to analyze this quantity. To find a more mathematically tractable form, note that, to a first-order approximation,

$$\sqrt{D+E} \approx \sqrt{D} + \frac{E}{2\sqrt{D}}, \quad \text{if } |E| << D.$$

Hence,

$$\text{ISER}_{\sqrt{\Psi}}(t) \approx \frac{2D(t)}{|E(t)|}, \quad \text{if } |E(t)| << D(t). \quad (45)$$

Thus, if by ISER_{Ψ} we denote the ISER at the output of Ψ,

$$\text{ISER}_{\Psi} >> 1 \Rightarrow \text{ISER}_{\sqrt{\Psi}} \approx 2 \cdot \text{ISER}_{\Psi}. \quad (46)$$

For example, if $\mu_b \approx 1$, $\omega_c/\omega_a = 10$ and $\kappa = 0.1$, then $\text{ISER}_{\Psi}(t) \geq 675$ and $\text{ISER}_{\sqrt{\Psi}}(t) \geq 1350$.

Concluding, we note that the envelope tracking abilities of $\sqrt{\Psi}$ also apply for certain signals $a(t)$ that are not band limited or have an infinite mean spectral value μ_a, in which cases Theorem 1 does not apply. For example, consider the case of an AM signal with exponential amplitude signal $a(t) = e^{rt}$ where r is any real number. Then (5) implies that $\sqrt{\Psi}$ can estimate the envelope with zero error. Another example is the case of a linear envelope $a(t) = st + c$, $t \geq 0$, where $s, c > 0$. Then since

$$\Psi[st + c] = s^2 \quad (47)$$

it follows from (11) that, for $t \geq 0$,

$$\sqrt{\Psi[(st+c)\cos(\omega_c t + \theta)]} \approx \omega_c(st + c), \quad \text{if } (s/c)^2 << \omega_c^2. \quad (48)$$

B. Discrete-Time AM

Consider the discrete operator Ψ applied to a general discrete-time real-valued AM signal $a(n)\cos(\Omega_c n + \theta)$, where $\Omega_c \in (0, \pi)$ is the discrete-time carrier frequency (in radians per sample). By (8) and (7) we have

$$\Psi[a(n)\cos(\Omega_c n + \theta)] = \underbrace{a^2(n)\sin^2(\Omega_c)}_{D(n)} + \underbrace{\Psi(a(n))[\cos^2(\Omega_c n + \theta) - \sin^2(\Omega_c)]}_{E(n)}. \quad (49)$$

For estimating the envelope $|a(n)|$, the desired energy term in the output of Ψ in $D(n)$. The remaining term $E(n)$ is viewed as an approximation error bounded by

$$|E(n)| \leq |\Psi(a(n))|. \quad (50)$$

Ψ can approximately track the envelope squared, if the instantaneous SER

$$\text{ISER}(n) = \frac{D(n)}{|E(n)|} \geq \frac{[\sin(\Omega_c)a_{\min}]^2}{\Psi(a)_{\max}} \quad (51)$$

is >> 1 for all n. This will be true if its lowest bound is >> 1. Hence, if $a_{\min} > 0$,

$$\Psi[a(n)\cos(\Omega_c n + \theta)] \approx [\sin(\Omega_c)a(n)]^2, \quad \text{if } \Psi(a)_{\max} << \sin^2(\Omega_c)a_{\min}^2. \quad (52)$$

Alternatively, for example if $a_{\min} = 0$, we can request as a milder condition that the average[3] SER

$$\text{ASER} = \frac{D_{\text{ave}}}{|E|_{\text{ave}}} \geq \frac{[\sin(\Omega_c)a_{\text{rms}}]^2}{|\Psi(a)|_{\text{ave}}} \quad (53)$$

be >> 1 over the time interval corresponding to the averaging operation. In this case

$$\Psi[a(n)\cos(\Omega_c n + \theta)] \approx_{\text{ave}} [\sin(\Omega_c)a(n)]^2, \quad \text{if } |\Psi(a)|_{\text{ave}} << [\sin(\Omega_c)a_{\text{rms}}]^2. \quad (54)$$

Next we provide some results concerning the maximum or mean absolute value of discrete-time signals, their differences, and their outputs from the energy operator. This analysis leads to finding upper bounds for the approximation error and lower bounds for the SER's. It is then simple to find sufficient conditions under which the SER's are much greater than unity and hence the above approximations are useful.

First note that if $x(n)$ is an aperiodic signal with Fourier transform $X(\Omega)$, then by using Parseval's theorem we obtain

$$\sum_{n=-\infty}^{\infty} \Psi[x(n)] = \frac{2}{\pi}\int_0^{\pi} \sin^2(\Omega)|X(\Omega)|^2 \, d\Omega. \quad (55)$$

Similarly, for a periodic signal $x(n) = \sum_{k=0}^{N-1} \alpha_k e^{j2\pi kn/N}$ with period N,

$$\frac{1}{N}\sum_{n=0}^{N-1} \Psi[x(n)] = 2\sum_{k=0}^{N-1} \sin^2\left(\frac{2\pi k}{N}\right)|\alpha_k|^2. \quad (56)$$

For any signal $x(n)$,

$$|x(n)| \leq x_{\max} \leq M_x \quad (57)$$

where

$$M_x \triangleq \begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^{\Omega_x} |X(\Omega)| \, d\Omega, & \text{if } x \text{ is aperiodic} \\ \displaystyle\sum_{k=0}^{N-1} |\alpha_k|, & \text{if } x(n) = \displaystyle\sum_{k=0}^{N=1} \alpha_k e^{j2\pi kn/N}. \end{cases}$$

[3]If a discrete-time signal $x(n)$ is aperiodic, then we define $x_{\text{ave}} = \Sigma_{n=-\infty}^{\infty} x(n)$. If x is periodic with period N, or if we analyze an arbitrary signal x only over a finite index interval $[0, N-1]$, then its average value is defined as $x_{\text{ave}} = (1/N)\Sigma_{n=1}^{N-1} x(n)$. In each case the rms value of x is defined as $x_{\text{rms}} = \sqrt{(x^2)_{\text{ave}}}$.

Further, $x_{\max} = M_x$ if x has linear Fourier phase; i.e., if x is aperiodic,

$$X(\Omega) = \pm|X(\Omega)|e^{-j\Omega n_0} \Rightarrow x_{\max} = M_x \tag{58}$$

or for periodic x

$$x(n) = \pm \sum_{k=0}^{\lfloor N/2 \rfloor} |A_k| \cos\left(\frac{2\pi k(n - n_0)}{N}\right) \Rightarrow$$
$$x_{\max} = M_x \tag{59}$$

where n_0 and $|A_k|$ are arbitrary. The results (57)–(59) are proved by using the same approach as for the proofs of (16)–(18) and discrete- instead of continuous-time Fourier transforms.

Proposition 2: Let $x(n)$ be a real-valued discrete-time signal that is band limited with highest frequency Ω_x, i.e., $X(\Omega) = 0$ for $\Omega_x < |\Omega| \leq \pi$. Then

$$|\underbrace{x(n) - x(n-1)}_{= x'(n)}| \leq 2 \sin(\Omega_x/2)M_x \tag{60}$$

$$|\underbrace{x(n) - 2x(n-1) + x(n-2)}_{= x''(n) = x'(n) - x'(n-1)}| \leq 4 \sin^2(\Omega_x/2)M_x \tag{61}$$

$$|\Psi(x(n))| \leq 8 \sin^2(\Omega_x/2)M_x^2 \tag{62}$$

$$\Psi[x(n)]_{\text{ave}} \leq 2[\sin(\Omega_x)x_{\text{rms}}]^2. \tag{63}$$

Proof: To prove (60) note that

$$|x'(n)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)(1 - e^{-j\Omega})e^{j\Omega n}\, d\Omega \right|$$
$$\leq \frac{1}{2\pi} \int_{-\Omega_x}^{\Omega_x} |X(\Omega)| \underbrace{|1 - e^{-j\Omega}|}_{2|\sin(\Omega/2)|}\, d\Omega$$
$$\leq 2 \sin(\Omega_x/2) \left(\frac{1}{\pi} \int_0^{\Omega_x} |X(\Omega)|\, d\Omega\right). \tag{64}$$

Now by (60) and since $|X'(\Omega)| = |2 \sin(\Omega/2)X(\Omega)|$,

$$|x''(n)| \leq 2 \sin(\Omega_x/2)M_{x'} \leq 4 \sin^2(\Omega_x/2)M_x$$

which proves (61). Finally, note that $\Psi[x(n-1)] = (x'(n))^2 - x(n)x''(n)$; hence

$$|\Psi(x(n-1))| \leq (x'_{\max})^2 + (xx'')_{\max}$$
$$\leq 8 \sin^2(\Omega_x/2)M_x^2.$$

This completes the proof of (62), because n is arbitrary. Finally, (63) results from (55) or (56) and the fact that $X(\Omega) = 0$ for $\Omega_x \leq |\Omega| \leq \pi$. Q.E.D.

The previous proposition has prepared the ground for finding the following bounds for the approximation error when the energy operator estimates the envelope term in the discrete AM signal.

Theorem 3: Consider a discrete-time AM signal $x(n) = a(n) \cos(\Omega_c n + \theta)$, where $a(n)$ is real valued and band limited with highest frequency $\Omega_a \in (0, \pi)$. Let $E(n) = \Psi[x(n)] - \sin^2(\Omega_c)a^2(n)$ be the approximation error. Then: (a)

$$|E(n)| \leq 8 \sin^2(\Omega_a/2)M_a^2. \tag{65}$$

If $\Psi[a(n)] \geq 0$ for all n, the average SER in the output of $\Psi(x)$ has the lower bound

$$\text{ASER} \geq \frac{1}{2}\left[\frac{\sin(\Omega_c)}{\sin(\Omega_a)}\right]^2. \tag{66}$$

(b) If $a(n) = \cos(\Omega_a n)$, then $|E(n)| \leq \sin^2(\Omega_a)$.

Proof: (a) follows from (50), (62), and (63). (b) follows from (50) and the fact that $\Psi[\cos(\Omega_a n)] = \sin^2(\Omega_a)$. Q.E.D.

Theorem 3 implies that $\sqrt{\Psi}$ can track the bandlimited envelope of discrete AM signals with very small mean absolute error provided that $2 \sin^2(\Omega_a)/\sin^2(\Omega_c) \ll 1$:

$$\sqrt{\Psi[a(n)\cos(\Omega_c n + \theta)]} \approx_{\text{ave}} |\sin(\Omega_c)a(n)|,$$
$$\text{if } 2\sin^2(\Omega_a)/\sin^2(\Omega_c) \ll 1. \tag{67}$$

Figure 1(a) shows an example with $a(n) = \cos(\Omega_a n)$, $\Omega_c = \pi/5$ and $\Omega_a = \pi/100$. For this example, Theorem 3 predicts that the average SER is ≥ 175 in the output of Ψ; the actual average SER was 494. (For simplicity, in this paper all the measured SERs have been rounded to integer values.) In the output of $\sqrt{\Psi}$ the average SER was 348. Finally, note that in taking the square root of $\Psi[x(n)]$ we implicitly assume that the latter is nonnegative for all n; sufficient conditions for this are provided in Section VI.

Next we focus on AM/WC signals.

Theorem 4: Consider a discrete-time AM/WC signal $x(n) = a(n) \cos(\Omega_c n + \theta)$, where $a(n) = 1 + \kappa b(n)$, $b(n)$ is real valued and band limited with highest frequency $\Omega_a \in (0, \pi)$, $|b(n)| \leq 1$, and $0 < \kappa < 1$. Let $E(n) = \Psi[x(n)] - \sin^2(\Omega_c)a^2(n)$ be the approximation error. Then: (a)

$$|E(n)| \leq 4\kappa \sin^2(\Omega_a/2)(M_b + 2\kappa M_b^2). \tag{68}$$

The instantaneous SER in the output of $\Psi(x)$ has the following lower bound:

$$\text{ISER}(n) \geq \left(\frac{\sin(\Omega_c)}{2\sin(\Omega_a/2)}\right)^2 \left[\frac{(1-\kappa)^2}{\kappa(M_b + 2\kappa M_b^2)}\right]. \tag{69}$$

(b) Over any finite index interval $[0, N-1]$ on which $\Sigma_{n=0}^{N-1} b(n) = 0$,

$$\text{ASER} \geq \left(\frac{\sin(\Omega_c)}{2\sin(\Omega_a/2)}\right)^2 \left[\frac{1 + (\kappa b_{\text{rms}})^2}{\kappa(M_b + 2\kappa M_b^2)}\right]. \tag{70}$$

If $b(n)$ is also periodic with period N and $\Psi(b) \geq 0$,

$$\text{ASER} \geq \frac{\sin^2(\Omega_c)[1 + (\kappa b_{\text{rms}})^2]}{2[\kappa b_{\text{rms}} \sin(\Omega_a)]^2 + 4\kappa b_{\text{rms}} \sin^2(\Omega_a/2)} \tag{71}$$

$$\geq \left(\frac{\sin(\Omega_c)}{2\sin(\Omega_a/2)}\right)^2 \left(\frac{1}{\kappa + 2\kappa^2}\right). \tag{72}$$

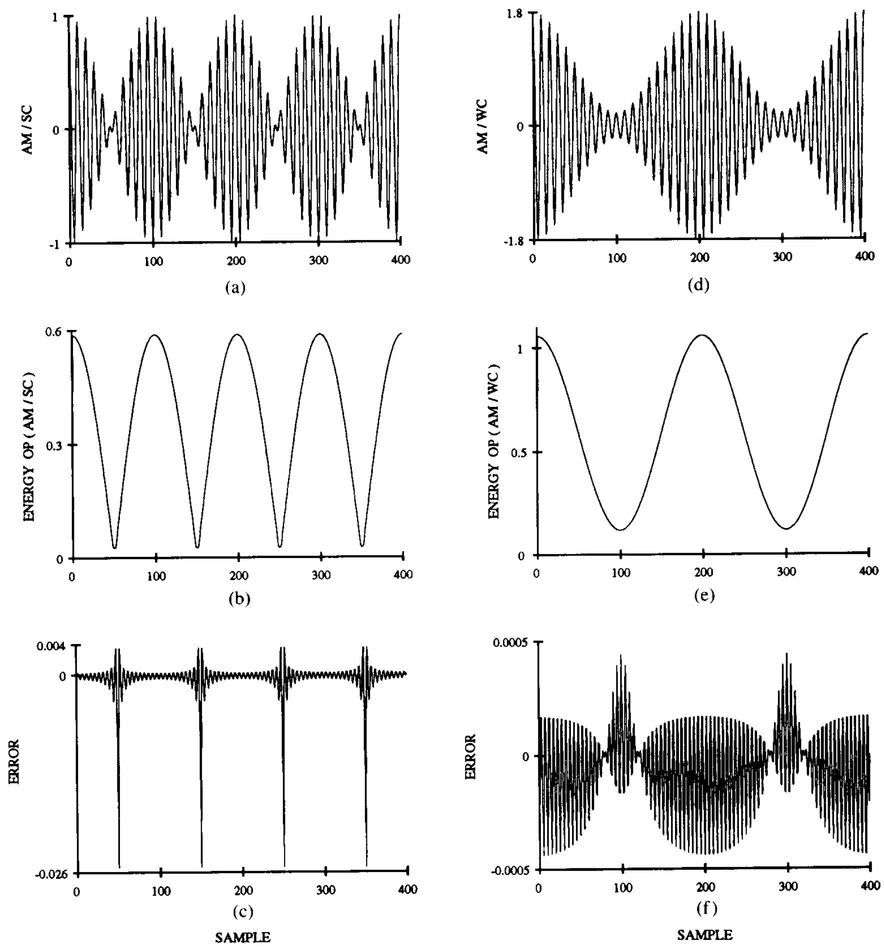

Fig. 1. Envelope detection by the energy operator Ψ in AM signals $a(n)\cos(\Omega_c n)$, where $\Omega_c = \pi/5$. (a) AM/SC signal with $a(n) = \cos(\pi n/100) = a_1(n)$. (b) $\sqrt{\Psi}$ applied on (a). (c) Difference between $|\sin(\Omega_c)a_1(n)|$ and the signal in (b). (d) AM/WC signal with $a(n) = 1 + 0.8\cos(\pi n/100) = a_2(n)$. (e) $\sqrt{\Psi}$ applied on (d). (f) Difference between $|\sin(\Omega_c)a_2(n)|$ and the signal in (e).

(c) If $b(n) = \cos(\Omega_a n)$,

$$|E(n)| \le \kappa^2 \sin^2(\Omega_a) + 4\kappa \sin^2(\Omega_a/2) \quad (73)$$

$$\text{ISER}(n) \ge \frac{\sin^2(\Omega_c)(1-\kappa)^2}{\kappa^2 \sin^2(\Omega_a) + 4\kappa\sin^2(\Omega_a/2)} \quad (74)$$

Proof: (a) First note the general property

$$\Psi[1 + \kappa b(n)] = \kappa^2 \Psi[b(n)] - \underbrace{\kappa[b(n+1) - 2b(n) + b(n-1)]}_{=b''(n+1)}. \quad (75)$$

Then (68) and (69) follow from (50), (51), (61), and (62) because

$$|\Psi(1 + \kappa b(n))| \le 4\kappa \sin^2(\Omega_a/2)(2\kappa M_b^2 + M_b) \quad (76)$$

(b) By (68),

$$|E|_{\text{ave}} = \frac{1}{N}\sum_{n=0}^{N-1} |E(n)| \le E_{\max} \le 4\kappa\sin^2(\Omega_a/2)(2\kappa M_b^2 + M_b). \quad (77)$$

Further, since $b_{\text{ave}} = 0$,

$$D_{\text{ave}} = \frac{\sin^2(\Omega_c)}{N}\sum_{n=0}^{N-1}[1 + \kappa b(n)]^2 = \sin^2(\Omega_c)[1 + (\kappa b_{\text{rms}})^2]. \quad (78)$$

The two previous results prove (70).

For periodic $b(n)$ we have $|b''|_{\text{ave}} \le 4\sin^2(\Omega_a/2) b_{\text{rms}}$. Then, by (75), (63), and the assumption $\Psi(b) \ge 0$,

$$|E|_{\text{ave}} \le 2[\kappa b_{\text{rms}} \sin(\Omega_a)]^2 + 4\kappa b_{\text{rms}} \sin^2(\Omega_a/2). \quad (79)$$

This and (78) prove (71), from which (72) follows since $b_{\text{rms}} \le 1$ and $\sin^2(\Omega) \le 4\sin^2(\Omega_a/2)$.

(c) Note that (75) yields

$$\Psi[1 + \kappa\cos(\Omega_a n)] = \kappa^2\sin^2(\Omega_a) + 4\kappa\sin^2(\Omega_a/2)\cos(\Omega_a n). \quad (80)$$

Then (73) and (74) follow because

$$|\Psi(1 + \kappa\cos(\Omega_a n))| \le \kappa^2\sin^2(\Omega_a) + 4\kappa\sin^2(\Omega_a/2). \quad (81)$$

Q.E.D.

By Theorem 4, assuming that $b(n)$ is zero mean and is periodic or has $M_b \approx 1$, we obtain

$$\sqrt{\Psi[(1 + \kappa b(n)) \cos(\Omega_c n + \theta)]} \approx_{\text{ave}} \sin(\Omega_c)[1 + \kappa b(n)] \tag{82}$$

if $4 \sin^2(\Omega_a/2)(\kappa + 2\kappa^2) << \sin^2(\Omega_c)$.

Fig. 1(d) shows an AM/WC signal and Fig. 1(e) shows the estimated envelope. The average SER in the output of Ψ was 1922, and the instantaneous $\text{ISER}_\Psi(n)$ ranged in [134, 19865]. This is consistent with Theorem 4 which predicts that ASER $\geq$ 383 and $\text{ISER}_{\min} \geq 10$. Further, in the output of $\sqrt{\Psi}$ we measured an average SER of 3359 and an instantaneous $\text{ISER}_{\sqrt{\Psi}}(n)$ in the range [267, 39730]; note that $\text{ISER}_{\sqrt{\Psi}}(n) \approx 2 \cdot \text{ISER}_\Psi(n)$ as predicted by (46). Thus, despite the small (carrier-to-information bandwidth) ratio $\Omega_c/\Omega_a = 20$ and the large AM index $\kappa = 0.8$, the operator $\sqrt{\Psi}$ performed quite well in estimating the envelope with an average relative error of 0.03%.

IV. Frequency Modulation (FM)

A. Continuous-Time FM

Consider the general FM signal

$$\cos[\phi(t)] = \cos\left[\omega_c t + \omega_m \int_0^t q(\tau)\, d\tau + \theta\right]$$

where ω_c is the carrier frequency, $q(t)$ is the FM modulating (or information) signal, $\phi(t) = \int_0^t \omega_i(\tau)d\tau + \theta$ is the phase signal, $\theta = \phi(0)$ is an arbitrary phase offset, and the instantaneous angular frequency is defined as

$$\omega_i(t) \triangleq \frac{d\phi(t)}{dt} = \omega_c + \omega_m q(t).$$

Henceforth, we shall always assume that

$$-1 \leq q(t) \leq 1 \quad \forall t; \qquad 0 < \omega_m < \omega_c$$

and ω_m is the maximum deviation of ω_i from ω_c. Hence, for all t,

$$0 < \omega_c - \omega_m \leq \omega_i(t) \leq \omega_c + \omega_m < 2\omega_c.$$

Given $\omega_i(t) = \dot{\phi}(t)$ and assuming that $q(t)$ achieves both its extreme values -1 and 1 within the analysis time interval, we can obtain ω_c, ω_m, $q(t)$ as follows:

$$\omega_c = \frac{(\omega_i)_{\min} + (\omega_i)_{\max}}{2}$$

$\omega_m = (\omega_i)_{\max} - \omega_c$, and $q(t) = [\omega_i(t) - \omega_c]/\omega_m$.

Applying Ψ to the FM signal yields

$$\Psi(\cos[\phi(t)]) = \underbrace{[\dot{\phi}(t)]^2}_{D(t)} + \underbrace{\ddot{\phi}(t)\frac{\sin[2\phi(t)]}{2}}_{E(t)}. \tag{83}$$

Our goal is to use Ψ for frequency demodulation, i.e., to estimate the instantaneous frequency $\omega_i(t)$, from which then the modulating signal $q(t)$ can be directly extracted. Hence, the desired energy term in (83) is $D = \omega_i^2$, whereas E is the approximation error bounded as

$$|E(t)| \leq \frac{\omega_m|\dot{q}(t)|}{2} \tag{84}$$

because $\ddot{\phi}(t) = \omega_m \dot{q}(t)$. A sufficient condition to ensure that $\Psi[\cos(\phi)] \approx \omega_i^2$ is to demand that the instantaneous SER

$$\text{ISER}(t) = \frac{D(t)}{|E(t)|} \geq \frac{2(\omega_c - \omega_m)^2}{\omega_m \dot{q}_{\max}} \tag{85}$$

be >> 1 for all t; then,

$$\Psi\left[\cos\left(\int_0^t \omega_i(\tau)\, d\tau\right)\right] \approx [\omega_i(t)]^2, \quad \text{if } \omega_m \dot{q}_{\max} << 2(\omega_c - \omega_m)^2. \tag{86}$$

Alternatively, for certain applications it may be sufficient to demand an average SER

$$\text{ASER} = \frac{D_{\text{ave}}}{|E|_{\text{ave}}} \geq \frac{2(\omega_i^2)_{\text{ave}}}{\omega_m |\dot{q}|_{\text{ave}}} \tag{87}$$

much greater than unity. The next theorem provides an upper bound for the absolute approximation error and lower bounds for the SER's. Henceforth, the FM index will be denoted by $\beta = \omega_m/\omega_f$.

Theorem 5: Assume that the real-valued FM information signal $q(t)$ is band limited with highest frequency ω_f and that its spectral moments $\mu_{q,0}$ and $\mu_{q,1}$ are finite. Then: (a) the approximation error $E = \Psi[\cos(\phi)] - \omega_i^2$ has the upper bound

$$|E(t)| \leq 0.5\omega_m \omega_f \mu_q. \tag{88}$$

The instantaneous SER at the output of Ψ has the lower bound

$$\text{ISER}(t) \geq \frac{2(\omega_c - \omega_m)^2}{\omega_m \omega_f \mu_q} \tag{89}$$

(b) Over any finite interval [0, T] on which $\int_0^T q(t)\, dt = 0$, the average SER has the lower bound

$$\text{ASER} \geq \frac{2[\omega_c^2 + (\omega_m q_{\text{rms}})^2]}{\omega_m \omega_f \mu_q} \tag{90}$$

If $q(t)$ is also periodic with period T, then

$$\text{ASER} \geq \frac{2[\omega_c^2 + (\omega_m q_{\text{rms}})^2]}{\omega_m \omega_f q_{\text{rms}}} \geq \frac{2}{\beta}\left(\frac{\omega_c}{\omega_f}\right)^2. \tag{91}$$

Proof: (a) Equations (88) and (89) follow from (84), (85), and (19) since $\dot{q}_{\max} \leq \omega_f \mu_q$.

(b) Over any finite interval [0, T] on which $q_{\text{ave}} = 0$,

$$(\omega_i^2)_{\text{ave}} = \frac{1}{T}\int_0^T [\omega_c + \omega_m q(t)]^2\, dt = \omega_c^2 + \omega_m^2 (q_{\text{rms}})^2. \tag{92}$$

In addition, $|\dot{q}|_{\text{ave}} \leq \dot{q}_{\max} \leq \omega_f \mu_q$. By combining these two inequalities, (87) yields (90).

If $q(t) = \Sigma_k \alpha_k e^{jk\omega_0 t}$ is also periodic with period $T = 2\pi/\omega_0$, then by the Cauchy–Schwarz inequality,

$$\frac{1}{T}\int_0^T |\dot{q}(t)|\, dt \le \sqrt{\frac{1}{T}\int_0^T |\dot{q}(t)|^2\, dt} = \sqrt{\omega_0^2 \sum_{k=-\omega_f/\omega_0}^{\omega_f/\omega_0} k^2|\alpha_k|^2} \le \sqrt{\frac{\omega_f^2}{T}\int_0^T |q(t)|^2\, dt} \le \omega_f q_{\text{rms}}.$$

Thus, for periodic q, $|\dot{q}|_{\text{ave}} \le \omega_f q_{\text{rms}}$. This proves (91); the second lower bound results because $q_{\text{rms}} \le 1$.

Q.E.D.

Assume now that ω_f, $\omega_m < \omega_c$. Then, if the FM information signal q is zero mean and is periodic or has $\mu_q \approx 1$, Theorem 5 guarantees that the average SER will be $>> 1$ under any of the following three realistic conditions: i) The bandwidth ω_f of the transmitted information is much smaller than the carrier frequency ω_c. ii) The frequency deviation ω_m is much smaller than ω_c. iii) The modulation index $\beta << 2$, which is true for all narrowband FM systems. (Both i) and ii) are standard assumptions in FM broadcasting systems [13].) Hence,

$$\Psi\left(\cos\left[\int_0^t \omega_i(\tau)\, d\tau\right]\right) \approx_{\text{ave}} [\omega_i(t)]^2, \quad \text{if } \omega_m\omega_f << 2\omega_c^2. \tag{93}$$

Alternatively, for any band limited $q(t)$ that satisfies the above assumptions (i)–(ii) and has $\mu_q \approx 1$ (which is true with equality if q has linear phase), then, by Theorem 5, the instantaneous SER will also be $>> 1$ and

$$\sqrt{\Psi\left(\cos\left[\int_0^t \omega_i(\tau)\, d\tau\right]\right)} \approx \omega_i(t), \quad \text{if } \omega_m\omega_f << 2(\omega_c - \omega_m)^2 \tag{94}$$

A special class of FM signals for which Theorem 5 does not apply is the class of FM/linear (chirp) signals, whose instantaneous frequency varies linearly:

$$\omega_i(t) = \omega_c + \omega_m\left(\frac{2t}{T} - 1\right), \qquad t \in [0, T]. \tag{95}$$

In this case we can still apply the approximation result (86) and obtain

$$\sqrt{\Psi\left[\cos\left(\omega_c t + \omega_m\left(\frac{t^2}{T} - t\right) + \theta\right)\right]} \approx \omega_c + \omega_m\left(\frac{2t}{T} - 1\right), \qquad \text{if } \frac{\omega_m}{T} << (\omega_c - \omega_m)^2. \tag{96}$$

Thus, $\sqrt{\Psi}$ can track the instantaneous frequency as long as the latter does not change much (i.e., small ω_m) or too fast (i.e., small $1/T$) compared to the carrier ω_c.

B. Discrete-Time FM

Consider a discrete-time FM signal

$$\cos[\phi(n)] = \cos\left[\Omega_c n + \Omega_m \int_0^n q(m)\, dm + \theta\right]. \tag{97}$$

We define its instantaneous angular frequency by

$$\Omega_i(n) \triangleq \frac{d\phi(n)}{dn} = \Omega_c + \Omega_m q(n). \tag{98}$$

Both the differentiation d/dn and the integration $\int dm$ in the above two definitions treat the integer time indices n and m symbolically as continuous variables. Note that the continuous-time angular frequencies ω_c, ω_m, and ω_i (in radians/second) have been replaced by their discrete-time counterparts Ω_c, Ω_m, and Ω_i (in radians/sample), which are assumed to be in $(0, \pi)$. We also assume that $|q(n)| \le 1$ for all n and $0 < \Omega_c \pm \Omega_m < \pi$.

In the discrete-time FM model (97) the modulating signal $q(n)$ is assumed to be a mathematical function with a known computable integral. For simplicity, during the rest of this section we will focus only on two special cases corresponding to $q(n)$ being either a cosine or a linear trend. Thus, consider first an FM/cosine signal

$$x(n) = \cos\underbrace{[\Omega_c n + \beta\sin(\Omega_f n) + \theta]}_{\phi(n)} \tag{99}$$

where $\beta = \Omega_m/\Omega_f$. The instantaneous frequency varies sinusoidally as

$$\Omega_i(n) = \Omega_c + \Omega_m\cos(\Omega_f n).$$

For applying Ψ to $x(n)$ note that if we set

$$A = \Omega_c n + \beta\cos(\Omega_f)\sin(\Omega_f n) + \theta$$
$$B = \Omega_c + \beta\sin(\Omega_f)\cos(\Omega_f n)$$

then

$$x(n+1)x(n-1) = \frac{\cos(2A) + \cos(2B)}{2} = \cos^2(A) - \sin^2(B).$$

Assume now that Ω_f is small; i.e.,

$$\Omega_f << 1 \Rightarrow \cos(\Omega_f) \approx 1 \quad \text{and} \quad \sin(\Omega_f) \approx \Omega_f. \tag{100}$$

For example, the approximation in (100) incurs an error $< 1\%$ for $\Omega_f \le \pi/50$. Then, $A \approx \phi(n)$, $B \approx \Omega_i(n)$, and

$$\Psi[\cos(\Omega_c n + \beta\sin(\Omega_f n) + \theta)] \approx \sin^2[\Omega_c + \Omega_m\cos(\Omega_f n)], \qquad \text{if } \Omega_f << 1. \tag{101}$$

Thus for discrete FM/cosine input signals, Ψ followed by a square root and $\sin^{-1}(\cdot)$ operation can approximately track their instantaneous frequency.

An alternative set of conditions for Ψ to track well the instantaneous frequency can be derived by bounding the approximation error. Namely, the approximation error $E = \Psi[\cos(\phi)] - \sin^2(\Omega_i)$ is equal to

$$\begin{aligned} E &= \cos^2(\phi) - \sin^2(\Omega_i) - \cos^2(A) + \sin^2(B) \\ &= [\cos(2\phi) + \cos(2\Omega_i) - \cos(2A) - \cos(2B)]/2 \\ &= \sin(A+\phi)\sin(A-\phi) + \sin(B+\Omega_i) \\ &\quad \cdot \sin(B-\Omega_i). \end{aligned} \tag{102}$$

The instantaneous SER in the approximation in (101) has the general lower bound:

$$\text{ISER}(n) = \frac{\sin^2[\Omega_i(n)]}{|E(n)|} \geq \frac{[\sin(\Omega_i)]^2_{\min}}{E_{\max}} \tag{103}$$

where

$$[\sin(\Omega_i)]_{\min} = \begin{cases} \sin(\Omega_c - \Omega_m), & \text{if } \Omega_c \leq \dfrac{\pi}{2} \\ \sin(\Omega_c + \Omega_m), & \text{if } \Omega_c > \dfrac{\pi}{2}. \end{cases} \tag{104}$$

Next we find a more specific lower bound for the instantaneous SER.

Proposition 3: Consider a discrete-time FM/cosine signal with $\Omega_f \leq \pi/4$. If

$$\beta \leq \frac{\pi}{2[1 - \sin(\Omega_f)]} \leq \frac{\pi}{2 - \sqrt{2}} \tag{105}$$

the instantaneous SER in the approximation in (101) has the following bound:

$$\text{ISER}(n) \geq \frac{[\sin(\Omega_i)]^2_{\min}}{\beta}. \tag{106}$$

Proof: If

$$\gamma_1 = \beta[1 - \cos(\Omega_f)], \qquad \gamma_2 = \beta[1 - \sin(\Omega_f)]$$

assumption (105) implies that both angles γ_1 and γ_2 are in the interval $[0, \pi/2]$. (Note that the first bound of β in (105) varies with Ω_f and assumes its maximum value of $\pi/(2 - \sqrt{2}) = 5.36$ when $\Omega_f = \pi/4$.) Now, by (102), $|E| \leq |\sin(A - \phi)| + |\sin(B - \Omega_i)|$, and hence $|E| \leq \sin(\gamma_1) + \sin(\gamma_2)$. This implies that $|E| \leq \gamma_1 + \gamma_2$, since $\sin(\theta) \leq \theta$ for any $\theta \in [0, \pi/2]$. Hence,

$$|E| \leq \beta\left[2 - \sqrt{2}\sin\left(\Omega_f + \frac{\pi}{4}\right)\right] \leq \beta$$

since $\sin(\Omega_f + \pi/4) \geq 1/\sqrt{2}$. This result and (103) complete the proof. Q.E.D.

Note that the lower bound (106) for the instantaneous SER will be $>> 1$ only for narrow-band FM/cosine signals, because $\beta << [\sin(\Omega_i)]^2_{\min}$ implies that $\beta << 1$.

Figs. 2(a) and (d) show two FM/cosine signals, the first with 100% modulation (i.e., $\Omega_m = \Omega_c$) and the second with 20% modulation (i.e., $\Omega_m = 0.2\Omega_c$). The corresponding outputs from the $\sqrt{\Psi}$ operator are shown in Figs. 2(b) and (e). The average SER's in the output of $\sqrt{\Psi}$ were 49 and 418, respectively. If we assume that the signals in Fig. 2 resulted from sampling continuous-time FM signals (with negligible aliasing), the lower bounds that Theorem 5 predicts for the two average SER's are 20 and 100, respectively. In the 20% modulation case we also measured $\text{ISER}(n) \in [169, 6650]$. Overall, the examples in Fig. 2 illustrate that the information signal is well tracked despite the fact that $\Omega_f/\Omega_c = 0.1$ is high compared to its usually low values in practical applications. Further, lowering Ω_m, i.e., the amount of frequency modulation, decreases the estimation error.

Consider now a discrete-time FM/linear signal over a finite time interval

$$y(n) = \cos\underbrace{\left[\Omega_c n + \Omega_m\left(\frac{n^2}{N} - n\right) + \theta\right]}_{\phi(n)}, \qquad n = 0, 1, \cdots, N \tag{107}$$

i.e., a "chirp" signal with linearly varying instantaneous frequency

$$\Omega_i(n) = \Omega_c + \Omega_m\left(\frac{2n}{N} - 1\right), \qquad n = 0, 1, \cdots, N.$$

In [2] it was shown that Ψ can approximately track the instantaneous frequency of the chirp signal if $\Omega_m/N << 1$. Next we follow a similar approach with the difference that all the approximations are left for the final stage so that the approximation error can be quantified. Given that

$$\begin{aligned} &y(n+1)y(n-1) \\ &\quad = \frac{1}{2}\left[\cos\left(2\phi(n) + \frac{2\Omega_m}{N}\right) + \cos(2\Omega_i(n))\right] \\ &\quad = \cos^2[\phi(n)] - \sin^2[\Omega_i(n)] \\ &\qquad + \frac{1}{2}\left[\cos\left(2\phi(n) + \frac{2\Omega_m}{N}\right) - \cos(2\phi(n))\right] \end{aligned}$$

it follows that

$$\begin{aligned} &\Psi[\cos(\phi(n))] \\ &\quad = \underbrace{\sin^2[\Omega_i(n)]}_{D(n)} + \underbrace{\sin\left[2\phi(n) + \frac{\Omega_m}{N}\right]\sin\left(\frac{\Omega_m}{N}\right)}_{E(n)}. \end{aligned} \tag{108}$$

Since $|E(n)| \leq \sin(\Omega_m/N)$, we have the approximation

$$\begin{aligned} &\Psi\left[\cos\left(\Omega_c n + \Omega_m\left(\frac{n^2}{N} - n\right) + \theta\right)\right] \\ &\quad \approx \sin^2\left[\Omega_c + \Omega_m\left(\frac{2n}{N} - 1\right)\right], \\ &\qquad \cdot \text{if } \sin\left(\frac{\Omega_m}{N}\right) << [\sin(\Omega_i)]^2_{\min} \end{aligned} \tag{109}$$

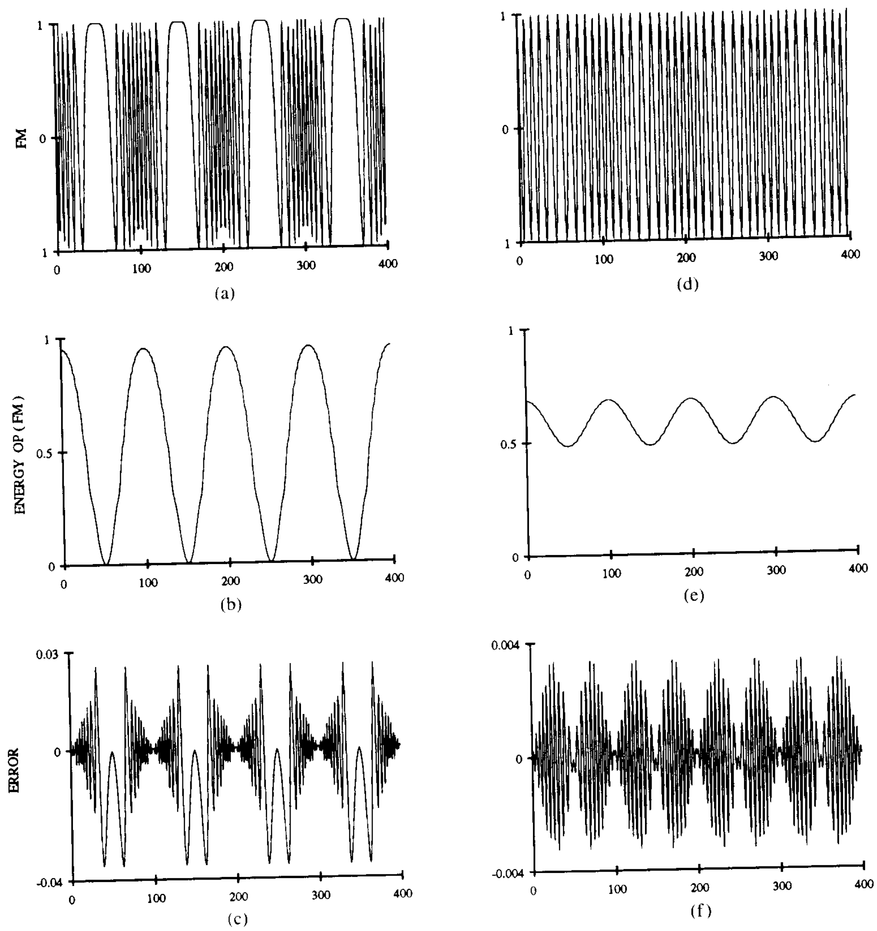

Fig. 2. Detection of instantaneous frequency by the energy operator Ψ in FM signals $\cos [\Omega_c n + (\Omega_m/\Omega_f) \sin (\Omega_f n)]$, where $\Omega_c = \pi/5$ and $\Omega_f = \pi/50$. (a) FM signal with $\Omega_m = \Omega_c$; $\Omega_i(n) = f_1(n)$. (b) $\sqrt{\Psi}$ applied on (a). (c) Difference between $|\sin (f_1(n))|$ and the signal in (b). (d) FM signal with $\Omega_m = 0.2\Omega_c$; $\Omega_i(n) = f_2(n)$. (e) $\sqrt{\Psi}$ applied on (d). (f) Difference between $|\sin (f_2(n))|$ and the signal in (e).

with an instantaneous SER

$$\mathrm{ISER}(n) \geq \frac{D_{\min}}{E_{\max}} \geq \frac{[\sin (\Omega_i)]^2_{\min}}{\sin (\Omega_m/N)}. \tag{110}$$

V. Amplitude and Frequency Modulation (AM-FM)

A. Continuous-Time AM-FM

The signal

$$a(t) \cos [\phi(t)] = a(t) \cos \left(\omega_c t + \omega_m \int_0^t q(\tau)\, d\tau + \theta \right) \tag{111}$$

is a cosine with both a time-varying amplitude $a(t)$ and a time-varying instantaneous frequency $\omega_i(t) = \dot{\phi}(t)$. It can be viewed either as a general FM signal whose amplitude varies like the envelope of some AM signal, or as a general AM signal whose instantaneous frequency is not constant but varies according to some FM information signal $q(t)$. We have used these types of combined amplitude and frequency modulation signals in modeling speech resonances [5]-[8]. We henceforth call (111) an AM-FM signal. Obviously, AM-FM signals include as special cases the AM and FM signals. Applying the operator Ψ yields

$$\Psi[a \cos (\phi)] = \underbrace{(a\dot{\phi})^2}_{D(t)} + \underbrace{a^2\ddot{\phi} \sin (2\phi)/2 + \cos^2 (\phi)\Psi(a)}_{E(t)}. \tag{112}$$

The desired energy term is $D = (a\dot{\phi})^2$. If we consider the approximation $\Psi[a \cos (\phi)] \approx (a\dot{\phi})^2$, then the error E is bounded as

$$|E(t)| \leq |\Psi(a(t))| + 0.5[a^2(t)\ddot{\phi}(t)] \leq \Psi(a)_{\max} + 0.5\omega_m a^2_{\max}\dot{q}_{\max}. \tag{113}$$

Let now $a(t)$ and $q(t)$ be band limited with corresponding highest frequencies $\omega_a < \omega_c$ and $\omega_f < \omega_c$. If $\Psi(a) \geq 0$, an upper bound for the mean absolute error is

$$|E|_{\text{ave}} \leq (2\omega_a^2 + 0.5\omega_m\omega_f\mu_q)(a_{\text{rms}})^2. \qquad (114)$$

Further, since $D_{\text{ave}} \geq (\omega_c - \omega_m)^2(a_{\text{rms}})^2$, the average SER of the approximation will have the lower bound

$$\text{ASER} \geq \frac{D_{\text{ave}}}{|E|_{\text{ave}}} \geq \frac{(\omega_c - \omega_m)^2}{2\omega_a^2 + 0.5\omega_m\omega_f\mu_q}. \qquad (115)$$

Thus, if $2\omega_a^2 + 0.5\omega_m\omega_f\mu_q << (\omega_c - \omega_m)^2$, then

$$\sqrt{\Psi\left[a(t)\cos\left(\int_0^t \omega_i(\tau)\,d\tau\right)\right]} \approx_{\text{ave}} |a(t)|\omega_i(t). \qquad (116)$$

Further, if the AM part is with carrier, i.e., if $a(t) = 1 + \kappa b(t) > 0$ with $\kappa < 1$, then we can also find a nonzero lower bound for the instantaneous SER. Specifically, from (113), (37), (23), and (19) we obtain

$$|E(t)| \leq \kappa\omega_a^2(\mu_b + 2\kappa\mu_b^2) + 0.5(1 + \kappa)^2\omega_m\omega_f\mu_q \qquad (117)$$

from which it follows that

$$\text{ISER}(t) \geq \frac{D_{\min}}{E_{\max}} \geq \frac{(1 - \kappa)^2(\omega_c - \omega_m)^2}{\kappa\omega_a^2(\mu_b + 2\kappa\mu_b^2) + 0.5(1 + \kappa)^2\omega_m\omega_f\mu_q}. \qquad (118)$$

Assuming that $\mu_b \approx 1$ and $\mu_q \approx 1$ (which are true with equality if b and q have linear phases, for example if they are single cosines), this SER will be $>> 1$ if $\omega_a, \omega_f, \omega_m << \omega_c$ and $\kappa << 1$; then

$$\sqrt{\Psi\left[(1 + \kappa b(t))\cos\left(\omega_c t + \omega_m\int_0^t q(\tau)\,d\tau + \theta\right)\right]} \approx [1 + \kappa b(t)][\omega_c + \omega_m q(t)]. \qquad (119)$$

Thus, under realistic assumptions, when $\sqrt{\Psi}$ is applied to an AM–FM signal, it yields the product of two components: the FM instantaneous frequency $\omega_i(t)$ and the AM envelope $|a(t)|$. If $\omega_m << \kappa\omega_c$, the $\omega_i(t)$ variations in the operator's output have much smaller amplitude than that of $|a(t)|$, and AM dominates over FM; i.e., the $\sqrt{\Psi}$ output follows the AM envelope. In contrast, FM dominates over AM if $\kappa << \omega_m/\omega_c$. Examples of these cases are shown in Section V-B.

Finally, if so desirable, it is possible to separate the amplitude from the frequency component in the output of $\sqrt{\Psi}$ applied to an AM–FM signal x via an algorithm which we have developed in [7], [8]. This algorithm is very efficient because it uses only a simple nonlinear combination of two energy signals, $\Psi(x)$ and $\Psi(\dot{x})$, to separately estimate the amplitude envelope and instantaneous frequency of x.

B. Discrete-Time AM–FM

Consider a real-valued discrete-time AM–FM signal

$$a(n)\cos[\phi(n)] = a(n)\cos\left(\Omega_c n + \Omega_m\int_0^n q(m)\,dm + \theta\right) \qquad (120)$$

with time-varying amplitude envelope $|a(n)|$ and instantaneous frequency $\Omega_i(n) = d\phi/dn = \Omega_c + \Omega_m q(n)$. For any a and ϕ,

$$\Psi[a(n)\cos\phi(n)] = a^2(n)\Psi[\cos\phi(n)] + \Psi[a(n)]\cos\phi(n-1)\cos\phi(n+1) \qquad (121)$$

$$= \underbrace{a^2(n)\sin^2\Omega_i(n)}_{D(n)} + \underbrace{a^2(n)[\Psi[\cos\phi(n)] - \sin^2\Omega_i(n)] + \Psi[a(n)]\cos\phi(n-1)\cdot\cos\phi(n+1)}_{E(n)}. \qquad (122)$$

Henceforth we assume that $\Omega_i \in (0, \pi)$ varies either sinusoidally, in which case $q(n) = \cos(\Omega_f n)$, or linearly, in which case $q(n) = 2n/N - 1$ for $n = 0, 1, \cdots, N$. In the cosine case, we assume that $\Omega_f << 1$. In the linear case, we assume that $\sin(\Omega_m/N) << [\sin(\Omega_i)]^2_{\min}$. These assumptions ensure that

$$\Psi[\cos(\phi(n))] \approx \sin^2[\Omega_i(n)]$$

and

$$E(n) \approx E'(n) = \Psi[a(n)]\cos[\phi(n-1)]\cos[\phi(n+1)].$$

Since $|E'(n)| \leq |\Psi(a(n))|$ for all n, we obtain the approximation

$$\Psi[a(n)\cos(\phi(n))] \approx_{\text{ave}} a^2(n)\sin^2[\Omega_i(n)], \quad \cdot \text{ if } |\Psi(a)|_{\text{ave}} << [a_{\text{rms}}\sin(\Omega_i)_{\min}]^2 \qquad (123)$$

with an average SER

$$\text{ASER} \geq \frac{(a_{\text{rms}})^2[\sin(\Omega_i)]^2_{\min}}{|\Psi(a)|_{\text{ave}}} \qquad (124)$$

$$\geq \frac{[\sin(\Omega_i)]^2_{\min}}{2\sin^2(\Omega_a)} \qquad (125)$$

where the second lower bound resulted by assuming that $a(n)$ is band limited with bandwidth Ω_a and that $\Psi(a)$ is nonnegative.

Further, if $a_{\min} > 0$, e.g., if $a(n) = 1 + \kappa b(n)$, where $b(n)$ is band limited with bandwidth Ω_a and $\kappa < 1$, then

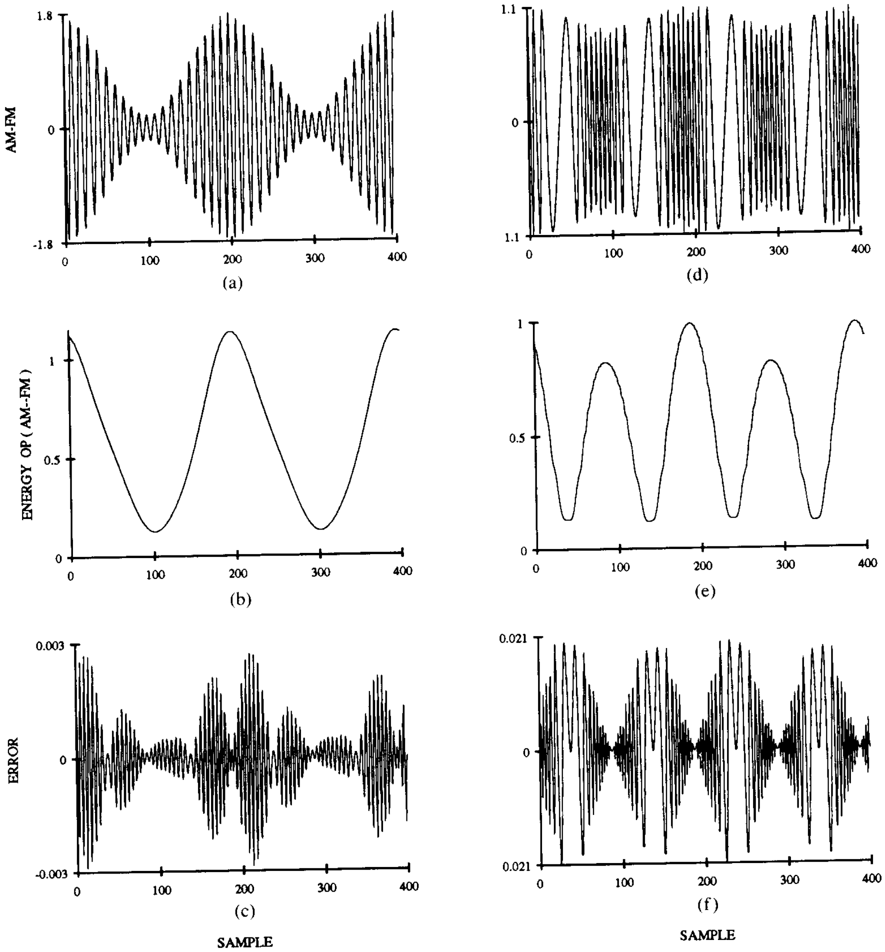

Fig. 3. Tracking abilities of the energy operator in AM/WC-FM signals $[1 + \kappa \cos(\Omega_a n)] \cos[\Omega_c n + (\Omega_m/\Omega_f)\sin(\Omega_f n + \pi/4)]$ with $\Omega_c = \pi/5$, $\Omega_a = \pi/100$, and $\Omega_f = \pi/50$. (a) AM/WC-FM signal with $\kappa = 0.8$ and $\Omega_m = 0.1\Omega_c$; $a(n) = a_1(n)$, $\Omega_i(n) = f_1(n)$. (b) $\sqrt{\Psi}$ applied on (a). (c) Difference signal between $|a_1(n) \sin(f_1(n))|$ and the signal in (b). (d) AM/WC-FM signal with $\kappa = 0.1$ and $\Omega_m = 0.8\Omega_c$; $a(n) = a_2(n)$, $\Omega_i(n) = f_2(n)$. (e) $\sqrt{\Psi}$ applied on (d). (f) Difference between $|a_2(n) \sin(f_2(n))|$ and the signal in (e).

we obtain the approximation

$$\Psi[a(n)\cos(\phi(n))] \approx a^2(n)\sin^2[\Omega_i(n)], \quad \text{if } \Psi(a)_{\max} \ll [a^2 \sin^2(\Omega_i)]_{\min} \tag{126}$$

with an instantaneous SER

$$\text{ISER}(n) \geq \frac{(1-\kappa)^2[\sin(\Omega_i)]^2_{\min}}{\Psi(a)_{\max}} \tag{127}$$

$$\geq \frac{(1-\kappa)^2[\sin(\Omega_i)]^2_{\min}}{4\kappa \sin^2(\Omega_a/2)(2\kappa M_b^2 + M_b)} \tag{128}$$

and an average SER (over any finite interval on which $b_{\text{ave}} = 0$)

$$\text{ASER} \geq \frac{[\sin(\Omega_i)]^2_{\min}}{8\kappa^2 \sin^2(\Omega_a/2) M_b^2}. \tag{129}$$

If $b(n) = \cos(\Omega_a n)$, it follows from (127) and (81) that

$$\text{ISER}(n) \geq \frac{(1-\kappa)^2[\sin(\Omega_i)]^2_{\min}}{\kappa^2 \sin^2(\Omega_a) + 4\kappa \sin^2(\Omega_a/2)}. \tag{130}$$

The above tracking of the squared product of the instantaneous amplitude and (sine of) frequency signals will incur a very small relative error as long as the instantaneous or average SER's are >> 1. Further, by (46), if $\text{ISER}_{\min}$ >> 1, then the instantaneous SER at the output of $\sqrt{\Psi}$ will be approximately twice the SER at the output of Ψ.

Figs. 3(a) and (d) show two AM/WC-FM signals with cosines as modulating signals and with the following two combinations of AM and FM amounts: i) 80% AM ($\kappa = 0.8$) and 10% FM ($\Omega_m/\Omega_c = 0.1$); ii) 10% AM and 80% FM. The corresponding outputs from the $\sqrt{\Psi}$ operator are shown in Figs. 3(b) and (e). The estimation error signals, i.e., the differences between the ideally desired outputs

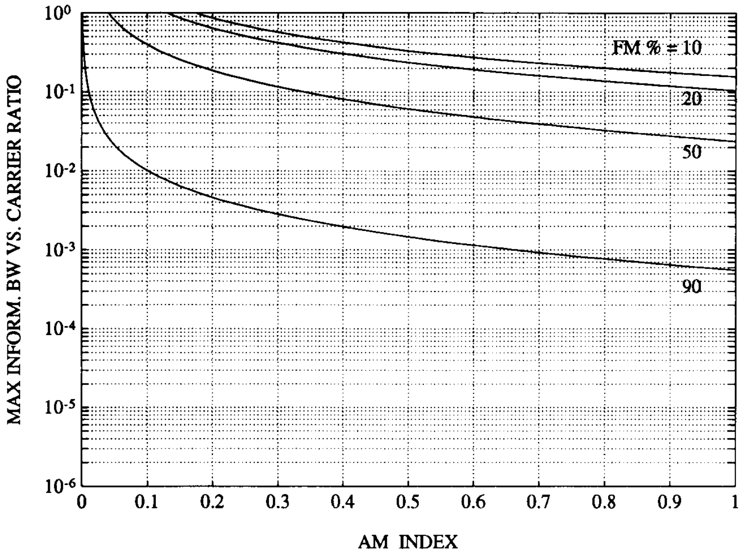

Fig. 4. Maximum allowable ratio ω_a/ω_c of the information signals' bandwidth versus the carrier so that the continuous energy operator Ψ is guaranteed to yield a nonnegative energy output when applied to AM-FM signals of the form $x(t) = [1 + \kappa \cos(\omega_a t)] \cos[\omega_c t + (\omega_m/\omega_a) \sin(\omega_a t)]$. The maximum ratio ω_a/ω_c, within (0, 1], is shown as a function of the AM index κ for different FM amounts $100(\omega_m/\omega_c)\%$.

$|a(n)| \sin[\Omega_i(n)]$ and the real outputs from the $\sqrt{\Psi}$ are shown in Figs. 3(c) and (f). For case i) we measured at the output of Ψ an ASER = 429, $\text{ISER}_{\min}$ = 134 and $\text{ISER}_{\max} \sim O(10^5)$. For the parameter values in Fig. 3(a), our theoretical results predict the lower bounds $\text{ISER}_{\min} \geq 8$ and ASER ≥ 227. In the output of $\sqrt{\Psi}$ we found that the actual values of the instantaneous and average SER's were approximately twice larger than their counterparts at the output of Ψ. For case ii), due to the very large amount of FM, the average SER at the output of $\sqrt{\Psi}$ dropped to 75 whereas $\text{ISER}_{\min} = 9$ and $\text{ISER}_{\max} \sim O(10^4)$. These examples illustrate that applying $\sqrt{\Psi}$ to discrete-time AM-FM signals can approximately track the product of the AM envelope and the (sine of the) FM instantaneous frequency, with a relatively small error even in cases with extreme amounts of AM or FM.

VI. Positivity of Energy Operators

For the validity of the approximate results derived in this paper it is assumed that we deal only with AM-FM signals $x(t) = a(t) \cos[\phi(t)]$ as in (111) for which $\Psi[x(t)] \geq 0$ for all t. From (112) it follows that, if $a(t) = 1 + \kappa b(t)$ with $|b(t)| \leq 1$ and $\kappa < 1$, a sufficient condition for the nonnegativity of $\Psi(x)$ is $E_{\max} \leq D_{\min}$, i.e.,

$$\Psi(a)_{\max} + 0.5\omega_m(1 + \kappa)^2 \dot{q}_{\max} \leq (1 - \kappa)^2(\omega_c - \omega_m)^2. \tag{131}$$

As an example, if $b(t) = \cos(\omega_a t + \theta)$ and $q(t) = \cos(\omega_f t)$, then by (44) the above condition becomes

$$0.5\omega_m\omega_f(1 + \kappa)^2 + \kappa(1 + \kappa)\omega_a^2 \leq (1 - \kappa)^2(\omega_c - \omega_m)^2. \tag{132}$$

Let us further assume that $\omega_a = \omega_f$ and define

$$\frac{\omega_a}{\omega_c} = \frac{\omega_f}{\omega_c} = r, \qquad \lambda = \frac{\omega_m}{\omega_c}$$

where r is the ratio of the information signal's bandwidth versus the carrier, and λ is the FM modulation depth. Then, for given amounts of AM and FM, i.e., for each combination of $(\kappa, \lambda) \in (0, 1)^2$, the allowable range for the ratio r that guarantees $\Psi[x(t)] \geq 0$ is

$$0 < r \leq \frac{\sqrt{\dfrac{\lambda^2}{4} + 4\kappa \dfrac{(1-\kappa)^2(1-\lambda)^2}{(1+\kappa)^2}} - \dfrac{\lambda}{2}}{2\kappa} = r_m.$$

As Fig. 4 shows, for very large AM and FM amounts $\approx$ 50% the maximum (r_m) allowable value of r in the interval (0, 1] is 0.05, which is relatively large. Conversely, for very large r such as 0.5, the signal can still have large amounts of AM and FM, e.g., 20%, and still yield a nonnegative energy output.

For an alternative condition, note that if the amplitude signal a is such that $\Psi[a(t)] \geq 0$ for all t, then a simpler condition to guarantee nonnegativity of $\Psi[a \cos(\phi)]$ is

$$\ddot{\phi}_{\max} \leq 2(\omega_c - \omega_m)^2. \tag{133}$$

If x has only FM, i.e., if a is constant, and q has linear Fourier phase and a bandwidth ω_f, then $\Psi(x) \geq 0$ if

$$\omega_m\omega_f \leq 2(\omega_c - \omega_m)^2. \tag{134}$$

This condition is true under very mild assumptions. For example, it is valid if $\omega_f/\omega_c \leq 0.1$ and $\omega_m/\omega_c \leq 0.8$.

If x has only AM, i.e., if $\omega_m = 0$, then $\Psi(a) \geq 0$ guarantees that $\Psi(x) \geq 0$. In general, there is a large class of signals a such that $\Psi(a) \geq 0$, as the following result indicates.

Proposition 4: (a) Any finite product of signals with nonnegative Ψ energy is also a signal with nonnegative energy; i.e., for all $n \geq 2$ and for all t,

$$\Psi[x_i(t)] \geq 0 \quad \forall i = 1, \cdots, n$$
$$\Rightarrow \Psi\left[\prod_{1 \leq i \leq n} x_i(t)\right] \geq 0. \tag{135}$$

(b) $\Psi(x)$ is nonnegative for any signal x that is a finite product of any signals from the following three classes: i) cosines of constant frequency, ii) real exponentials $e^{\sigma t}$, and iii) linear trends $st + c$.

Proof: (a) Since $\Psi(x_1x_2) = x_1^2\Psi(x_2) + x_2^2\Psi(x_1)$, (135) is true for $n = 2$. By induction on n, it can also be proven for any integer $n > 2$. (b) is a special case of (a) since $\Psi(\cos(\omega_c t)) = \omega_c^2$, $\Psi(e^{\sigma t}) = 0$, and $\Psi(st + c) = s^2$. Q.E.D.

Note also that, from the definition $\Psi(x) = (\dot{x})^2 - x\ddot{x}$, we can predict the positivity or negativity of $\Psi(x)$ at special time instants. Specifically, $\Psi[x(t_0)] \geq 0$ if t_0 is a zero crossing ($x = 0$) or inflection ($\ddot{x} = 0$) point. Also, $\Psi[x(t_0)] < 0$ if x has a positive local minimum ($x > 0$, $\dot{x} = 0$, $\ddot{x} > 0$) at t_0 or a negative local maximum.

In the discrete-time case, it is also assumed in this paper that we deal with AM–FM signals $x(n) = a(n)\cos[\phi(n)]$ as in (120) such that $\Psi[x(n)] \geq 0$ for all n. If $a(n) = 1 + \kappa b(n)$ with $|b(n)| \leq 1$ and $\kappa < 1$, then it follows from (122) that a sufficient condition for the nonnegativity of $\Psi(x)$ is $E_{\max} \leq D_{\min}$, i.e.,

$$\Psi(a)_{\max} + (1 + \kappa)^2[\Psi(\cos(\phi)) - \sin^2(\Omega_i)]_{\max} \leq (1 - \kappa)^2[\sin(\Omega_i)]^2_{\min}. \tag{136}$$

As discussed for the continuous-time case, there are many classes of discrete AM–FM signals that satisfy (136). For example, in the case of a cosine amplitude $b(n) = \cos(\Omega_a n)$ and a linear frequency $\Omega_i(n) = \Omega_c + \Omega_m(2n/N - 1)$, $n = 0, \cdots, N$, with $\Omega_c \leq \pi/2$, it follows from (136), (81), and (108) that $\Psi(x) \geq 0$ if

$$\kappa^2 \sin^2(\Omega_a) + 4\kappa \sin^2(\Omega_a/2) + (1 + \kappa)^2 \sin\left(\frac{\Omega_m}{N}\right) \leq (1 - \kappa)^2 \sin^2(\Omega_c - \Omega_m). \tag{137}$$

For more general cases it suffices for the present work to say that, during most of our experiments with noiseless AM–FM and bandpass filtered speech signals, we have very rarely encountered a negative $\Psi[x(n)]$, and in most such cases the negative value appeared to be due to roundoff errors.

VII. Discussion

We have shown that the energy operator Ψ, followed by a square root operation, can approximately estimate the envelope of AM signals and the instantaneous frequency of FM signals. In both cases we have found upper bounds for the maximum and mean absolute value of the approximation errors. We have also shown that their magnitudes relative to the corresponding signal values are much smaller than unity under very general conditions for the modulating signals, namely, by assuming that the AM or FM information signals have a bandwidth much smaller than the carrier frequency and/or that they do not vary much in value with respect to the carrier. Our results apply to general AM signals both in continuous and discrete time, as well as to general continuous-time FM signals. For discrete-time FM signals we presented results in the simple cases where the modulating signal varies either sinusoidally or linearly; in [7] we have extended our discrete FM results to more general classes of signals that are finite linear combinations of cosines.

We have also analyzed general AM–FM signals $a(t)\cos(\int_0^t \omega_i(\tau)d\tau)$, for which we showed that (if the relative approximations errors involved in the AM and FM demodulation are << 1)

$$\sqrt{\Psi\left[a(t)\cos\left(\int_0^t \omega_i(\tau)d\tau\right)\right]} \approx |a(t)|\,\omega_i(t).$$

Thus the $\sqrt{\Psi}$ output is the product of two parts: the instantaneous frequency $\omega_i(t)$ and the amplitude envelope $|a(t)|$. This result generalizes the tracking ability of $\sqrt{\Psi}$, which for cosines $A\cos(\omega_0 t)$ yields $|A|\omega_0$, whereas for AM–FM signals the constant amplitude A and frequency ω_0 are replaced by the time-varying amplitude and instantaneous frequency.

Finally, all our results for continuous-time signals can be easily extended to incorporate any multiplicative constant amplitude $A \neq 1$ and/or any exponential factor e^{rt} in the input signal by just multiplying the energy operator's output with A^2e^{2rt}, because

$$\Psi_c[Ae^{rt}x(t)] = A^2e^{2rt}\Psi_c[x(t)].$$

Similarly, all our results for discrete-time signals can be easily extended to incorporate a constant amplitude $A \neq 1$ and/or an exponential factor r^n in the input signal by just multiplying the output of Ψ by A^2r^{2n}, since $\Psi_d[Ar^nx(n)] = A^2r^{2n}\Psi_d[x(n)]$.

A. Noise

Throughout all our analysis we assumed that the AM and FM signals are clean, i.e., do not contain noise. Next we provide some preliminary discussion for the case when the signal is corrupted by noise. Kaiser [2] showed that if $w(n)$ is a discrete-time zero-mean white noise sequence, then

$$\mathcal{E}\{\Psi[x(n) + w(n)]\} = \Psi[x(n)] + \sigma_w^2 \tag{138}$$

where $\mathcal{E}\{\cdot\}$ denotes statistical expectation and $\sigma_w^2 = \mathcal{E}\{w^2(n)\}$. We illustrate the effect of (138) in the case of signals $x(n)$ with a bandwidth $\Omega_x < \pi$ by comparing the signal-to-noise ratio (SNR), defined as the ratio of mean squared values, of the noisy input $x + w$ and output $\Psi(x$

$+ w)$. Thus, the SNR of the input is $\text{SNR}_I = (x_{\text{rms}})^2/\sigma_w^2$. Note that $\Psi(x + w)$ is equal to $\Psi(x)$ plus some noise terms and that the output of Ψ has the same dimension as the signal squared. Hence, the SNR of the output is $\text{SNR}_O = |\Psi(x)|_{\text{ave}}/\sigma_w^2$. If $\Psi(x) \geq 0$, it follows from (63) that

$$\text{SNR}_O \leq 2 \sin^2 (\Omega_x)\text{SNR}_I \tag{139}$$

with equality if $x(n) = A \cos (\Omega_x n)$. Thus, if the input is a cosine corrupted by noise and $2 \sin^2 (\Omega_x) \approx 1$, then the energy operator does not deteriorate the input SNR; it can even lead to an SNR improvement since the output SNR will be larger than the input SNR if $\pi/4 < \Omega_x < 3\pi/4$. There are several issues that arise concerning the implications of (138) and (139) for general AM or FM signals as well as possible ways of suppressing noise either in the input or in the output of Ψ. These issues are not addressed in this paper.

B. Postfiltering the Error

The approximation error signal E when Ψ is used to track the envelope of an AM signal $a(t) \cos (\omega_c t + \theta)$ has a low-pass and a high-pass component, i.e., by (9),

$$E(t) = \Psi(a(t)) \left[\frac{1}{2} + \frac{\cos (2\omega_c t + 2\theta)}{2}\right].$$

The high-pass error component $\Psi(a) \cos (2\omega_c t + 2\theta)$ can be effectively eliminated by a low-pass filtering in the output of Ψ. Thus, as observed in [11], the approximation error can be further reduced by about 50% if the output of Ψ is low-pass filtered to eliminate components around $2\omega_c$. Of course, the relative magnitude of this high-pass error component with respect to the desired term $a^2\omega_c^2$ is in the order of $O(\omega_a^2/\omega_c^2)$. Thus, for small ratios $\omega_a/\omega_c << 1$, the error is already negligible and hence a further low-pass filtering may be an unnecessary computational luxury.

For FM signals $\cos [\phi(t)]$, the approximation error when Ψ tracks the instantaneous frequency term $\omega_i^2(t)$ is, by (83), $E(t) = \ddot{\phi}(t) \sin [2\phi(t)]/2$. Hence, if ω_f is the bandwidth of $\omega_i(t)$, the spectrum of the error is approximately concentrated in the interval $[2\omega_c - 2\omega_m - 2\omega_f, 2\omega_c + 2\omega_m + 2\omega_f]$, whereas the desired term ω_i^2 has highest frequency $2\omega_f$. Thus, if the ratio $(\omega_m + 2\omega_f)/\omega_c$ is smaller than one, then elimination of the error component through low-pass postfiltering is possible, although a very small value of this ratio (which is common in realistic FM systems) guarantees that the error magnitude is negligible compared to the desired term.

Similar conclusions can be made for the general AM–FM case and for discrete-time signals, as has been experimentally found in [11].

C. Other Demodulation Approaches

A standard envelope estimation approach in AM systems [13] consists of passing the AM signal through a memoryless nonlinearity (e.g., a square law), followed by low-pass filtering. In these systems the estimation errors are introduced either via the imperfections of the low-pass filter that always has a nonideal cutoff frequency response or via the nonlinearities that may introduce distortions such as higher powers of the envelope. However, the energy operator approach does not require a low-pass filter. Using a low-pass filter at the output of $\sqrt{\Psi}$ may further reduce the approximation error, but it may not be needed since the error is already small if $\omega_a << \omega_c$ or $\kappa << 1$. Similar arguments apply for the FM case.

Given a real AM–FM signal $x(t) = a(t) \cos [\phi(t)]$, an alternative approach to estimate its envelope $|a(t)|$ and instantaneous frequency $\dot{\phi}(t)$ is to use the Hilbert transform $\hat{x}(t)$ of $x(t)$. The Hilbert transform approach can provide an envelope

$$r(t) = \sqrt{x^2(t) + \hat{x}^2(t)}$$

and an instantaneous frequency $\dot{\theta}(t)$, where $\theta(t) = \arctan [\hat{x}(t)/x(t)]$. Of course, there will be some error because $r(t)$ and $\dot{\theta}(t)$ will generally be different from their counterparts $|a(t)|$ and $\dot{\phi}(t)$ imposed by the true modulations in x. In [11] there is some work reported on comparing the Hilbert transform with the energy operator approach in [7], [8]. Experiments on N-sample discrete AM–FM signals indicate that when the ratio of carrier ω_c versus the information signals' bandwidths ω_a, ω_f is in the order of $O(10)$, then the discrete Hilbert transform (implemented via an FIR filter) can give a smaller error but at a computational complexity $O(N^2)$ which is higher than the very low $O(N)$ complexity of the energy operator approach. Decreasing the complexity of the Hilbert transform to an $O(N)$ by using a shorter impulse response makes its error larger than that of the energy operator. In addition, if the ratio of ω_c versus ω_a, ω_f is in the order of $O(100)$ or higher, then the energy operator yields a smaller error than the more complex Hilbert transform approach.

D. Applications

In proving that the energy operator can be used for tracking the envelope of AM signals or the instantaneous frequency of FM signals with negligible error we assumed that the carrier frequency ω_c is much larger than the bandwidth ω_a or ω_f of the AM or FM information signal and that the AM index κ or the FM depth ω_m/ω_c are much smaller than one. All these assumptions are very realistic and universally used by AM or FM communication systems. For example, commercial AM broadcasting systems use a carrier frequency in the range [550, 1600] kHz and the transmitted information is band limited to 5 kHz; hence, in this case the average relative error that Ψ would incur while tracking the envelope would be in the order of $\omega_a^2/\omega_c^2 < 10^{-4}$. Similarly in commercial FM broadcasting systems the carrier is in the range [88, 108] MHz, whereas the message bandwidth is 15 kHz, and the maximum frequency deviation does not exceed 75 kHz. This implies that using Ψ to track the instantaneous frequency would incur an average relative error in the order of $\omega_m\omega_f/\omega_c^2 < 10^{-6}$. Thus both in AM/WC and FM communications systems the energy operator is applicable since it can provide signal demodulation with very small error and can be implemented very simply.

Another very promising applications area for the results in this paper is the problem of tracking modulations in speech resonances. Motivated by several nonlinear and time-varying phenomena during speech production, we proposed in [5], [6] an AM–FM modulation model for speech signals by representing a single speech (formant) resonance as a damped AM–FM signal and the total short-time speech signal $S(t)$ within a pitch period as a sum of such AM–FM signals

$$S(t) = \sum_{k=1}^{K} a_k(t) \cos\left(\omega_{c,k}t + \omega_{m,k}\int_0^t q_k(\tau)d\tau + \theta_k\right)$$

where K is the number of speech formants, $\omega_{c,k}$ is the center value of the kth formant, $\omega_{m,k}$ is its maximum frequency deviation from $\omega_{c,k}$, $q_k(t)$ is the normalized frequency deviation signal, and $a_k(t) = e^{-\sigma_k t}A_k(t)$ is its time-varying amplitude that includes an exponential decay and a generally non-constant amplitude signal $A_k(t)$. The instantaneous value of the kth formant frequency is $\omega_{i,k}(t) = \omega_{c,k} + \omega_{m,k}q_k(t)$. In contrast to the above AM–FM model, the standard linear time-invariant (within a short-time frame) model for speech implies that each formant has no FM (i.e., $\omega_{m,k} = 0$) and no AM beyond the exponential damping (i.e., $A_k(t)$ is a constant). In our speech experiments [5]–[8] we have found that the energy operator applied to single speech resonances (extracted via bandpass filtering of speech) is very efficient in tracking modulation patterns with significant amounts of AM and FM. These time-varying instantaneous frequencies and amplitudes extracted from each speech resonance can be used for a variety of applications including speech coding, synthesis and recognition.

In another application, we have applied the energy operator to detection of weak acoustical events in an interfering AM–FM signal background [12]. Examples of such interference occur in an underwater environment (e.g., interference from biologics or active sonar) or in environments for mechanical system monitoring (e.g., interference from large rotating machinery). We have developed an AM–FM model for the narrowband background, and to detect acoustic signals in this background we applied the discrete energy operator. When the signal of interest does not satisfy the assumed AM–FM model, the operator's output is shown to produce a large deviation from the slowly-varying modulation components. One interpretation of this deviation is that the operator is performing a nonlinear short-time estimation of the AM–FM background; when the background deviates from the assumed model, the estimation error grows. In essence the energy operator senses the perturbations from the AM–FM background. Preliminary results indicate that this technique can detect short-duration signals in a slowly-varying AM–FM background. Robustness of the method in noise and the formulation of a "detection statistic" are now being investigated.

In conclusion, the instantaneous adapting nature of the energy operator and its ability to track amplitude and frequency modulations, as well as its low complexity, makes the energy operator a very useful tool for signal processing and its application to communications, speech analysis, and signal detection.

Acknowledgment

The authors wish to thank the anonymous reviewers whose constructive comments helped improve this paper. They thank A. Bovik of the University of Texas at Austin for many insightful discussions and for deriving a result regarding the expected value of the energy operator applied to random signals which inspired (21). They also thank A. Potamianos at Harvard University for performing the computer experiments on comparing the Hilbert transform and the energy operator.

References

[1] K. Chandrasekharan, *Classical Fourier Theorems.* Berlin: Springer-Verlag, 1989.
[2] J. F. Kaiser, "On a simple algorithm to calculate the 'energy' of a signal," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, Albuquerque, NM, Apr. 1990, pp. 381–384.
[3] J. F. Kaiser, "On Teager's energy algorithm and its generalization to continuous signals," in *Proc. 4th IEEE Digital Signal Processing Workshop*, Mohonk (New Paltz), NY, Sept. 1990.
[4] J. F. Kaiser, "On Teager's energy algorithm, its continuous and discrete forms, some of their properties, and their application to speech analysis," manuscript in preparation.
[5] P. Maragos, T. F. Quatieri, and J. F. Kaiser, "Detecting nonlinearities in speech using an energy operator," in *Proc. 4th IEEE Digital Signal Processing Workshop*, Mohonk (New Paltz), NY, Sept. 1990.
[6] P. Maragos, T. F. Quatieri, and J. F. Kaiser, "Speech nonlinearities, modulations, and energy operators," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, Toronto, Ontario, Canada, May 1991.
[7] P. Maragos, J. F. Kaiser, and T. F. Quatieri, "Energy separation in signal modulations with application to speech analysis," Tech. Rep. 91-17, Harvard Robotics Lab., Harvard Univ., Nov. 1991; also *IEEE Trans. Signal Processing*, submitted for publication.
[8] P. Maragos, J. F. Kaiser, and T. F. Quatieri, "On separating amplitude from frequency modulations using energy operators," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, San Francisco, CA, Mar. 1992.
[9] D. C. Miller, *Sound Waves, Their Shape and Speed.* New York: Macmillan, 1937, p. 49.
[10] A. Papoulis, *The Fourier Transform and Its Applications.* New York: McGraw-Hill, 1962.
[11] A. Potamianos and P. Maragos, "A comparison of the energy operator and Hilbert transform approaches to signal and speech demodulation," Tech. Rep., Harvard Robotics Lab, Harvard Univ., 1992.
[12] T. F. Quatieri, J. F. Kaiser, and P. Maragos, "Transient detection in AM-FM background using an energy operator," presented at the 1991 IEEE Underwater Acoustic Signal Processing Workshop, Univ. Rhode Island, Oct. 1991.
[13] M. Schwartz, *Information Transmission, Modulation and Noise.* New York: McGraw-Hill, 1980.
[14] H. M. Teager, "Some observations on oral air flow during phonation," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-28, no. 5, pp. 599–601, Oct. 1980.
[15] H. M. Teager and S. M. Teager, "A phenomenological model for vowel production in the vocal tract," in *Speech Sciences: Recent Advances*, R. G. Daniloff, Ed. San Diego, CA: College-Hill Press, 1983, pp. 73–109.
[16] H. M. Teager and S. M. Teager, "Evidence for nonlinear production mechanisms in the vocal tract," in *NATO Advanced Study Inst. Speech Production Speech Modelling*, Bonas, France, July 24, 1989, pp. 241–261, 1990.

Petros Maragos (S'81-M'85-SM'91) was born in Kalymnos, Greece, on November 4, 1957. He received the Diploma degree in electrical engineering from the National Technical University of Athens, Greece, in 1980, and the M.S.E.E. and Ph.D. degrees from the Georgia Institute of Technology, Atlanta, in 1982 and 1985, respectively.

In 1985 he joined the faculty of the Division of Applied Sciences at Harvard University, Cambridge, where he is currently an Associate Professor of Electrical Engineering. During the fall of 1992, he was a Visiting Professor at the National Technical University of Athens. His general research and teaching activities have been in the areas of signal processing, image processing and computer vision, and computer speech processing. His current research focuses on morphological image processing, nonlinear systems for speech processing, fractal signal analysis, and pattern recognition.

Dr. Maragos served as an Associate Editor for the IEEE TRANSACTIONS ON SIGNAL PROCESSING from 1989-1991. He is currently on the Editorial Board for the *Journal of Visual Communication and Image Representation*. In 1983, he received a Sigma Xi thesis research award for his work on image coding. In 1987, he received a National Science Foundation Presidential Young Investigator Award for his work in image and signal processing. He is also the recipient of the IEEE Acoustics, Speech, and Signal Processing Society's 1988 Paper Award for the paper "Morphological Filters," published in the society's transactions.

James F. Kaiser (S'50-A'52-SM'70-F'73) was born in Piqua, OH, in 1929. He received the E.E. degree from the University of Cincinnati, Cincinnati, OH, in 1952 and the S.M. and Sc.D. degrees in 1954 and 1959, respectively, from M.I.T., Cambridge, MA, all in electrical engineering.

Currently he is a Visiting Professor in the Department of Electrical and Computer Engineering at Rutgers University, Piscataway, NJ. He was formerly a Distinguished Member of the Technical Staff in the Speech and Image Processing Research Division of Bell Communications Research, Inc., which he joined in 1984 at its formation. Prior to that he was a Distinguished Member of the Technical Staff at Bell Laboratories, Murray Hill, NJ, for 25 years, where he worked in the areas of speech processing, system simulation, signal processing, computer graphics, and computer-aided design. He also serves as a consultant to industry. He is the author of more than 60 research papers and the coauthor and editor of seven books in the signal processing and automatic control areas.

Dr. Kaiser has served in a number of positions in both the IEEE Signal Processing Society and the IEEE Circuits and Systems Society, including Administrative Committee member, Secretary-Treasurer, Digital Signal Processing Technical Committee member, and Awards Board Chair and member. He has also served on the IEEE Press Board and as an ECPD Visitor. For six years he was Secretary of the COSINE Committee of the Commission on Education of the National Academy of Engineering. He is a member of Eta Kappa Nu, Tau Beta Pi, and Sigma Xi. He also received the Technical Achievement Award of the Signal Processing Society in 1977, its Meritorious Service Award in 1979, its Society Award in 1982, and the IEEE Centennial Medal in 1984. In 1970 he received the Distinguished Engineering Alumnus Award of the University of Cincinnati, College of Engineering, and in 1980 the Eta Kappa Nu Award of Merit, also by the University of Cincinnati. He is a Registered Engineer in Massachusetts, and a member of ASA, AAAS, EURASIP, and SIAM.

Thomas F. Quatieri (S'73-M'79-SM'87) was born in Somerville, MA, on January 31, 1952. He received the B.S. degree (*summa cum laude*) from Tufts University, Medford, MA, in 1973, and the S.M., E.E., and Sc.D. degrees from the Massachusetts Institute of Technology (M.I.T.), Cambridge, MA in 1975, 1977, and 1979, respectively.

In 1980, he joined the Sensor Processing Technology Group of M.I.T., Lincoln Laboratory, Lexington, MA, where he worked on problems in multidimensional digital signal processing and image processing. Since 1983 he has been a member of the Speech Systems Technology Group at Lincoln Laboratory where he has been involved in digital signal processing for speech coding and enhancement, underwater sound enhancement, and data communications. He has contributed many publications to journals and conference proceedings, written several patents, and coauthored chapters in two edited books: *Advanced Topics in Signal Processing* (Prentice-Hall, 1987) and *Advances in Speech Signal Processing* (Marcel Dekker, 1991). He is also the lecturer for the M.I.T. graduate course on *Digital Speech Processing*.

Dr. Quatieri received the 1982 Paper Award of the IEEE Acoustics, Speech, and Signal Processing Society for the paper, "Implementation of 2-D Digital Filters by Iterative Methods." In 1990, he received the IEEE Signal Processing Society's Senior Award for the paper "Speech Analysis/Synthesis Based on a Sinusoidal Representation," published in the IEEE TRANSACTIONS ON ACOUSTICS, SPEECH, AND SIGNAL PROCESSING. He is a member of the IEEE Digital Signal Processing Technical Committee, and since 1983 has served on the steering committee for the biannual Digital Signal Processing Workshop. He is also a member of Tau Beta Pi, Eta Kappa Nu, and Sigma Xi.