Higher Order Differential Energy Operators

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***Abstract*—Instantaneous signal operators $\Upsilon_k(x) = \dot{x}x^{(k-1)} - xx^{(k)}$ of integer orders k are proposed to measure the cross energy between a signal x and its derivatives. These higher order differential energy operators contain as a special case, for $k = 2$, the Teager–Kaiser operator. When applied to (possibly modulated) sinusoids, they yield several new energy measurements useful for parameter estimation or AM-FM demodulation. Applying them to sampled signals involves replacing derivatives with differences that lead to several useful discrete energy operators defined on an extremely short window of samples.**

I. Higher Order Energy Measurements

INSTANTANEOUS differences in the relative rate of change between two signals x, y can be measured via their Lie bracket

$$[x, y] \equiv \dot{x}y - x\dot{y}$$

because $[x, y]/xy = (\dot{x}/x) - (\dot{y}/y)$. Dots denote time derivatives. Note the antisymmetry $[x, y] = -[y, x]$. If $y = \dot{x}$, then $[x, y]$ becomes the continuous-time Teager–Kaiser energy operator [1], [2]

$$\Psi(x) \equiv (\dot{x})^2 - x\ddot{x} = [x, \dot{x}]$$

that has been used for tracking the energy of a source producing an oscillation [2], [1] and for signal and speech AM-FM demodulation [4], [5]. In the general case, if x and y represent displacements in some generalized motions, the quantity $[x, \dot{y}] = \dot{x}\dot{y} - x\ddot{y}$ has dimensions of energy (per unit mass), and hence, we may view it as a *'cross energy'* between x and y. This energy like quantity $\dot{x}\dot{y} - x\ddot{y}$ was used in [2], [3] to analyze the output $\Psi(x + y)$ of the energy operator applied to a sum of two signals.

In our work, we use the cross energy between a signal x and its higher order derivatives to develop higher order energy measurements. Specifically, we define the kth-order *differential energy operator (DEO)*

$$\Upsilon_k(x) \equiv [x, x^{(k-1)}] = \dot{x}x^{(k-1)} - xx^{(k)}, \quad k = 0, \pm 1, \pm 2, \cdots$$

as yielding the cross energy between a signal $x(t)$ and its $(k-1)$th derivative (or integral), where

$$x^{(k)}(t) \equiv \begin{cases} d^k x(t)/dt^k, & k \geq 1 \\ x(t), & k = 0 \\ \int_{-\infty}^{t} x^{(k+1)}(\tau)d\tau, & k \leq -1 \end{cases}$$

denotes a signal derivative for positive order k or an integral for k negative. Of practical current interest are the DEO's of positive orders. The second-order DEO Υ_2, measuring the energy of a harmonic oscillator producing a signal x, gives to Υ_k the name "energy," since it is identical to the standard energy operator Ψ. The zeroth-order operator is $\Upsilon_0(x) = \dot{x}\int x - x^2$; this latter expression was recognized in [3] as the negative of the energy of the signal integral. The first-order DEO yields zero for any signal. Two new and useful energy measurements are given by the third- and fourth-order DEO's:

$$\Upsilon_3(x) \equiv \dot{x}\ddot{x} - xx^{(3)} \quad , \quad \Upsilon_4(x) \equiv \dot{x}x^{(3)} - xx^{(4)}.$$

Note that (as also observed in [3])

$$\Upsilon_3(x) = \frac{d\Psi(x)}{dt} \quad , \quad \Upsilon_4(x) = \frac{d\Upsilon_3(x)}{dt} - \Psi(\dot{x}).$$

Hence, the third-order DEO Υ_3 is an *energy velocity* operator, whereas the fourth-order DEO Υ_4 has dimensions of *energy acceleration*. In general, the higher order operators can be generated by lower order operators with a two-step recursion:

$$\Upsilon_k(x) = \frac{d\Upsilon_{k-1}(x)}{dt} - \Upsilon_{k-2}(\dot{x}).$$

Finally, note that

$$\Upsilon_k(x + y) = \Upsilon_k(x) + \Upsilon_k(y) + [x, y^{(k-1)}] + [y, x^{(k-1)}].$$

When the energy operators Υ_k are applied to a sine wave, they yield products of powers of the amplitude and frequency. Specifically, the cosine

$$x(t) = A\cos(\omega t + \theta)$$

representing the response of an undamped harmonic oscillator satisfies the motion equation $\ddot{x} + \omega^2 x = 0$. This creates the energy recursion

$$E_k = -\omega^2 E_{k-2} \quad , \quad E_k \equiv \Upsilon_k[A\cos(\omega t + \theta)]$$

with initial conditions $E_0 = -A^2$ and $E_1 = 0$. Running this recursive equation in both forward and backward order index k yields

$$\Upsilon_k[A\cos(\omega t + \theta)] = \begin{cases} 0, & k = \pm 1, \pm 3, \pm 5, \cdots \\ (-1)^{1+\frac{k}{2}} A^2 \omega^k, & k = 0, \pm 2, \pm 4, \cdots \end{cases}$$

If the amplitude A and/or frequency ω of $x(t)$ are slowly time-varying, i.e., if x is an AM-FM signal, then the above energy equations are approximately valid provided that $A = A(t)$ and $\omega = \omega(t)$ do not vary too fast or too much with respect to the carrier frequency. Further, because $A^2\omega^k$ are lowpass signals, the above instantaneous energy measurements can be used for

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robust estimation of instantaneous amplitude and frequency in modulated sinusoids.

An application of the fourth-order DEO Υ_4, in conjunction with the standard energy operator $\Upsilon_2 \equiv \Psi$, is to estimate the amplitude and frequency of a (possibly modulated) sinusoid $x(t) = A\cos(\omega t + \theta)$:

$$\omega = \sqrt{\frac{-\Upsilon_4(x)}{\Upsilon_2(x)}} \quad , \quad |A| = \frac{\Upsilon_2(x)}{\sqrt{-\Upsilon_4(x)}}.$$

This is an energy separation algorithm, slightly different from the one in [5], which can also be used for AM-FM demodulation.

An application of the third-order DEO Υ_3 is to estimate the energy dissipation rate in damped oscillations. Namely, given a damped cosine, the damping factor can be found using Υ_3 and the energy operator. Thus, if $x(t) = Ae^{-rt}\cos(\omega t + \theta)$, $r > 0$, then

$$r = -\frac{\Upsilon_3(x)}{2\Upsilon_2(x)} = -\frac{1}{2}\frac{d\log\Upsilon_2(x)}{dt}.$$

II. Discrete-Time Operators

Applying the energy operators to sampled signals requires replacing derivatives with differences. This leads to a variety of discrete energy operators for each order k, because there are many different ways of discretizing derivatives. The simplest approach is to first discretize the Lie bracket by replacing derivatives with time shifts. Namely, replacing continuous-time signals $x(t)$ with sequences $x_n = x(nT)$ of their samples, also denoted as $x[n]$, and first-order derivatives $\dot{x}(t)$ with backward differences $\Delta_b x[n] = (x[n] - x[n-1])/T$ converts the continuous-time operator $[x, y](t)$ into the discrete-time operator

$$C(x[n], y[n]) \equiv x[n]y[n-1] - x[n-1]y[n]$$

where we henceforth assume $T = 1$. (Using symmetric differences $\Delta_s x[n] = (x[n+1] - x[n-1])/2$ to replace time derivatives yields a symmetric discrete operator equal to the average of C at two consecutive samples.) Using $y[n] = x[n+1]$ makes C identical to the discrete Teager–Kaiser energy operator [1]

$$\Psi(x[n]) \equiv x^2[n] - x[n-1]x[n+1] = C(x[n], x[n+1]).$$

Generalizing the above result by using $y[n] = x[n+k]$ in C leads us to develop discrete-time higher order energy measurements for a signal $x[n]$. For example, we define the kth-order discrete[1] energy operator

$$\begin{aligned}\Upsilon_k(x[n]) &\equiv C(x[n], x[n+k-1]) \quad , \quad k = 0, 1, 2, 3, \cdots \\ &= x[n]x[n+k-2] - x[n-1]x[n+k-1].\end{aligned}$$

For $k = 1$, we always get zero since $\Upsilon_1 \equiv 0$. For $k = 2$, we obtain the standard discrete energy operator $\Upsilon_2 \equiv \Psi$. For $k = 3$, we obtain an *asymmetric discrete energy velocity operator*

$$\Upsilon_3(x_n) \equiv x_n x_{n+1} - x_{n-1}x_{n+2}$$

whereas $k = 4$ yields a discrete energy acceleration operator:

$$\Upsilon_4(x_n) \equiv x_n x_{n+2} - x_{n-1}x_{n+3}.$$

Important aspects of each Υ_k are the length of its corresponding index window and its time alignment (a)symmetry. Next, we investigate these issues for $k = 3$. Since Υ_3 requires a four-sample moving window $[n-1, n+2]$, its output at the window's center occurs at the continuous time instant $t = (n + 0.5)T$. One simple approach to eliminate this time misalignment is to replace $\Upsilon_3(x_n)$ with its average over two consecutive samples and thus have a *symmetric* third-order energy operator

$$\Upsilon_{3s}(x_n) \equiv \frac{\Upsilon_3(x_n) + \Upsilon_3(x_{n-1})}{2}$$

with a five-sample window $[n-2, n+2]$.

Applying the operators Υ_k to discrete (possibly damped) cosines yields discrete energy equations

$$\Upsilon_k[Ar^n\cos(\Omega n + \theta)] = A^2 r^{2n+k-2}\sin(\Omega)\sin[(k-1)\Omega]$$

which are useful for parameter estimation in sinusoids. In addition, these energy equations hold approximately when the cosine has time-varying amplitude and frequency that do not vary too fast or too much with respect to the carrier, i.e., when the input is a sampled AM-FM signal. This allows us to find discrete AM-FM demodulation algorithms by combining the above energy equations of various orders. For example, by using Υ_2, Υ_3, and the undamped cosine energy equations $\Upsilon_k[A\cos(\Omega n + \theta)] = A^2\sin(\Omega)\sin[(k-1)\Omega)]$ for $k = 2, 3$, a discrete algorithm was proposed in [6] for instantaneous frequency tracking, which is closely related to the discrete energy separation algorithm in [5].

We conclude by noting that, all the above discrete higher order energy operators can be unified as special cases of a class of *quadratic energy operators* Q_{km}, or their weighted linear combinations, where

$$Q_{km}(x[n]) \equiv x[n]x[n+k] - x[n-m]x[n+k+m]$$

for $k = 0, 1, 2, \cdots$, $m = 1, 2, \cdots$. Similar operators have also been studied independently by Kaiser [7]. The class Q contains all the discrete higher order energy operators Υ_k since $Q_{k1} \equiv \Upsilon_{k+2}$; e.g., $Q_{01} \equiv \Psi$ and $Q_{11} \equiv \Upsilon_3$. For $k = 0$ the operators Q_{0m} can also be viewed as special cases of the class of quadratic detectors $\sum_m h_m x[n+m]x[n-m]$ proposed in [8]. The general operators Q_{km} provide some interesting energy equations:

$$Q_{km}[Ar^n\cos(\Omega n + \theta)] = A^2 r^{2n+k}\sin(m\Omega)\sin[(m+k)\Omega].$$

In addition, each Q_{km} can be generated recursively from operators of lower orders k, m.

[1] For notational simplicity, we use the same symbol for both the continuous- and discrete-time higher order operators Υ_k and the Teager–Kaiser energy operator Ψ since the input signal can reveal this aspect of the operator.

III. Alternative Discretizations

Instead of discretizing the Lie bracket and replacing derivatives with time shifts, an alternative approach to discretizing Υ_k is to replace each mth-order signal derivative involved in its expression with backward difference operators ${\Delta_b}^m = \Delta_b({\Delta_b}^{m-1})$ or symmetric differences ${\Delta_s}^m = \Delta_s({\Delta_s}^{m-1})$. For $k = 2$ the asymmetric difference yields a one-sample shifted version of the discrete energy operator Ψ, whereas the symmetric difference yields a three-point average of Ψ, as shown in [4]. For $k = 3$ using the Δ_b difference yields another asymmetric discrete energy velocity operator

$$\begin{aligned}\Upsilon_{3b}(x_n) &\equiv \Upsilon_3(x)|_{d^m/dt^m \mapsto {\Delta_b}^m}\ (nT)\\ &= 2\Upsilon_2(x_{n-1}) - \Upsilon_3(x_{n-2})\end{aligned}$$

which is computationally more complicated than Υ_3. The only slight advantage of Υ_{3b} over Υ_3 is that when the input is a discrete cosine $x_n = A\cos(\Omega n + \theta)$, it gives as output $4A^2\sin^2(\Omega)\sin^2(\Omega/2)$, which for $\Omega \ll 1$ is much closer to zero than the output $A^2\sin(\Omega)\sin(2\Omega)$ of Υ_3. Recall that the continuous-time third-order energy of a cosine is zero. Using the Δ_s difference for $k = 3$ yields another symmetric discrete energy velocity operator

$$\begin{aligned}&\Upsilon'_{3s}(x_n) \equiv \Upsilon_3(x)|_{d^m/dt^m \mapsto {\Delta_s}^m}\ (nT) =\\ &\tfrac{1}{8}(\Upsilon_3(x_{n+1}) + \Upsilon_3(x_n) - \Upsilon_3(x_{n-1}) - \Upsilon_3(x_{n-2}))\end{aligned}$$

which yields a zero when the input is a discrete cosine (consistent with the continuous-time result) but has a longer window than the symmetric operator Υ_{3s}, i.e., it needs a seven-sample window $[n-3, n+3]$. For $k > 3$, we get even more complicated expressions, using either the Δ_b or the Δ_s differences.

A final approach we have considered for discretizing the continuous Υ_k operators is to i) replace derivatives $x^{(k)}$ with differences ${\Delta_b}^k x$; and ii) shift the differences by any required number of samples so that the two terms $\dot{x}x^{(k-1)}$ and $xx^{(k)}$ are computed at the same time location after discretization. Odd-order derivatives are centered at time instants $(n \mp 0.5)T$; if k is even, this is balanced by the other odd derivative in the product that is centered at $(n \pm 0.5)T$. This approach yields the same discrete operator for $k = 2$ but creates alternative discrete Υ_k operators with interesting properties for $k > 2$. Thus, for $k = 2, 3, 4$, we obtain the alternative operators Υ_{ka}

$$\begin{aligned}\Upsilon_{2a}(x_n) &\equiv \Psi(x_n)\\ \Upsilon_{3a}(x_n) &\equiv \Psi(x_n) - \Psi(x_{n-1}) = \Delta_b\Upsilon_{2a}(x_n)\\ \Upsilon_{4a}(x_n) &\equiv [\Psi(x_{n+1}) - 2\Psi(x_n) + \Psi(x_{n-1})]\\ &\qquad - \Psi(x_n - x_{n-1})\\ &= \Delta_b\Upsilon_{3a}(x_{n+1}) - \Upsilon_{2a}(\Delta_b x_n).\end{aligned}$$

For example, to obtain $\Upsilon_{3a}(x_n)$ from $\Upsilon_3(x) \equiv \dot{x}\ddot{x} - xx^{(3)}$, the odd order derivatives $\dot{x}$, $x^{(3)}$ are discretized (using backward differences) and centered at time $(n - 0.5)T$, whereas the discretization of $\ddot{x}$ is centered at time nT (so that $\dot{x}\ddot{x}$ and $xx^{(3)}$ are computed at the same time instant). Note that Υ_{3a} yields zero when the input is a discrete cosine, as in the continuous-time case. The above discrete Υ_{ka} operators require a small window and satisfy recursive formulas of the same type as the recursion $\Upsilon_k(x) = d\Upsilon_{k-1}(x)/dt - \Upsilon_{k-2}(\dot{x})$ satisfied by their continuous counterparts (with derivatives mapped to differences Δ_b). This is their main advantage. In general, the best type of discretization of higher order energies depends on the specific application.

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