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Algebraic and PDE Approaches for Lattice Scale-Spaces with Global Constraints

PETROS MARAGOS
School of Electrical and Computer Engineering, National Technical University of Athens, Zografou, Athens 15773, Greece
maragos@cs.ntua.gr

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Abstract. This paper begins with analyzing the theoretical connections between levelings on lattices and scale-space erosions on reference semilattices. They both represent large classes of self-dual morphological operators that exhibit both local computation and global constraints. Such operators are useful in numerous image analysis and vision tasks including edge-preserving multiscale smoothing, image simplification, feature and object detection, segmentation, shape and motion analysis. Previous definitions and constructions of levelings were either discrete or continuous using a PDE. We bridge this gap by introducing generalized levelings based on triphase operators that switch among three phases, one of which is a global constraint. The triphase operators include as special cases useful classes of semilattice erosions. Algebraically, levelings are created as limits of iterated or multiscale triphase operators. The subclass of multiscale geodesic triphase operators obeys a semigroup, which we exploit to find PDEs that can generate geodesic levelings and continuous-scale semilattice erosions. We discuss theoretical aspects of these PDEs, propose discrete algorithms for their numerical solution which converge as iterations of triphase operators, and provide insights via image experiments.

Keywords: nonlinear scale-spaces, morphology, lattice operators, levelings, PDEs

1. Introduction

Nonlinear scale-space approaches that are based on morphological operators are useful for edge-preserving multiscale smoothing, image simplification, geometric feature detection, segmentation, shape, texture and motion analysis, and object recognition. The theory and implementations behind the standard multiscale morphological filters evolved first (Matheron, 1975; Serra, 1982; Maragos, 1989) from a geometric viewpoint that focused on shape-size analysis and a simple algebraic viewpoint that was based on set theory and min-max filtering. During the last decade both the algebraic and geometric aspects of morphology were generalized and improved, by extending its algebra using the theory of complete lattices (Serra, 1988; Heijmans, 1994) and by modeling the dynamics and geometry of multiscale morphology using partial differential equations (PDEs) and curve evolution (Alvarez et al., 1993; Brockett and Maragos, 1994; Sapiro et al., 1993). Openings and closings are the basic morphological smoothing filters. The simplest openings/closings, which are compositions of Minkowski flat erosions and dilations, preserve well the edges of the remaining image signal parts but may blur the boundaries of their supports at places where the structuring element cannot fit. A much more powerful class of filters are the *reconstruction* openings and closings which, starting from a *reference* image consisting of several parts and a *marker* (initial seed) inside some of these parts, can reconstruct whole objects with exact preservation of their boundaries and edges (Vincent, 1993; Salembier and Serra, 1995). In this reconstruction process they simplify the original image by completely eliminating smaller objects inside which the marker cannot fit. The reference image plays the role of a *global constraint*. One disadvantage of both the simple as well as the reconstruction openings/closings is that they are not self-dual and hence they treat asymmetrically the image foreground vs. background or the bright vs. dark objects. A recent solution to this asymmetry problem came from the development of a more general powerful class of self-dual morphological filters, the *levelings* introduced by Meyer (1998) and further studied in Matheron (1997) and Serra (2000), which include as special cases the reconstruction openings and closings. The levelings possess many useful algebraic scale-space properties, as explored in Meyer and Maragos (2000), which are best studied in a lattice framework. Further, they can be generated by the following nonlinear PDE[1] introduced in Maragos and Meyer (1999):

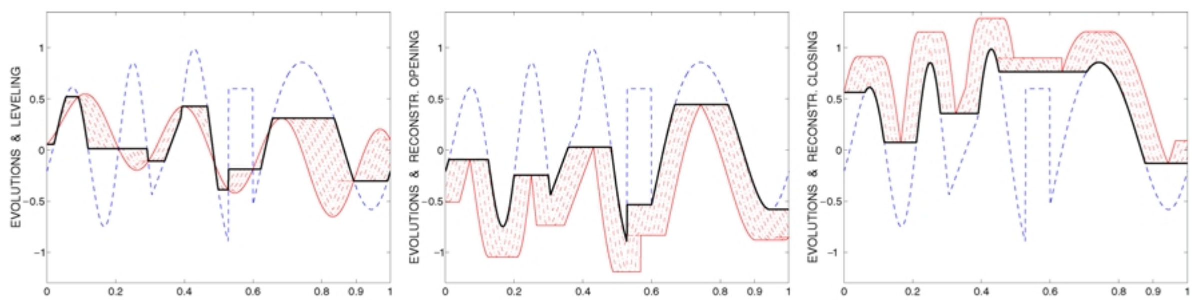
$$\partial u(x,y,t)/\partial t = -\mathrm{sign}(u-r)\|\nabla u\| \\ u(x,y,0) = f(x,y) \tag{1}$$

where $u(x,y,t)$ is the scale-space function, $r(x,y)$ is the reference image and $f(x,y)$ is a marker. At scale-space points where $u > r$ (resp., $u < r$), the above PDE generates multiscale erosions (resp., dilations) by disks. The leveling of r w.r.t. f is produced when $t \to \infty$. In Maragos and Meyer (1999) and Meyer and Maragos (2000) it was explained that, if $f \leq r$ (resp., $f \geq r$), the leveling is a reconstruction opening (resp., closing). Examples are shown in Fig. 1.

One way to construct multiscale levelings is to use a sequence of multiscale markers obtained from sampling a Gaussian scale-space. As shown in Fig. 2, the image edges and boundaries which have been blurred and shifted by the Gaussian scale-space are better preserved across scales by the multiscale levelings.

A relatively new algebraic approach to self-dual morphology was developed by Keshet (2000) and Heijmans and Keshet (2000, 2001) based not on complete lattices but on inf-semillatices. Specifically, by using self-dual partial orderings the image space becomes an inf-semilattice on which self-dual erosion operators can be defined that have many interesting properties and promising applications in nonlinear image analysis.

In this paper we develop theoretical connections between levelings on lattices and erosions on semilattices, both from an algebraic and a PDE viewpoint. We begin in Section 2 with a brief background discussion on multiscale operators defined on complete lattices and inf-semilattices. In Section 3 we introduce and analyze algebraically multiscale *triphase* operators (which switch among 3 different states, one state being a global constraint) whose special cases are reference semilattice erosions and whose limits are levelings. The semigroup of *geodesic* triphase operators is discovered. Afterwards, in Section 4 we model both geodesic levelings and semilattice erosions using PDEs. The main ingredient here is the leveling PDE which we prove it can generate both the multiscale geodesic operators and (after some modification) the translation-invariant semilattice self-dual erosions. Section 5 extends the PDE ideas to 2D images signals. In both Sections 4 and 5 we propose discrete numerical algorithms for solving the PDEs and prove their convergence using triphase operators. Concluding comments,



(a) (b) (c)

Figure 1. Evolutions of 1D leveling PDE $u_t = -\mathrm{sign}(u-r)\,|u_x|$ for 3 different markers $u(x,0) = f(x)$. Each figure shows the reference signal r (dash line), the marker f (thin solid line), its evolutions $u(x,t)$ (thin dashdot line) at $t = n25\Delta t$, $n = 1, 2, \ldots$, and the leveling $u(x,\infty)$ (thick solid line). The 3 markers f were: (a) Arbitrary. (b) An erosion of r minus a constant; hence, the leveling is a reconstruction opening. (c) A dilation of r plus a constant; hence the leveling is a reconstruction closing. ($\Delta x = 0.001$, $\Delta t = 0.0005$).

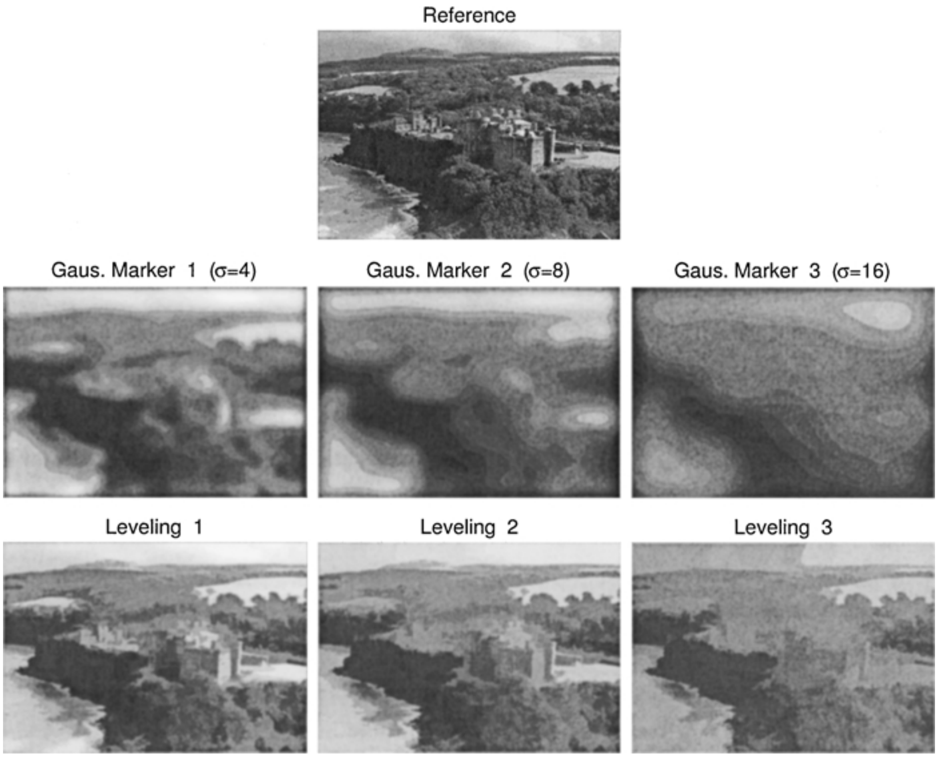


Figure 2. Multiscale image levelings. The markers were obtained by convolving the reference image with 2D Gaussians of standard deviations $\sigma_1 = 4$, $\sigma_2 = 8$, $\sigma_3 = 16$. At each scale σ_i as reference was used the leveling of the previous scale σ_{i-1}.

insights via experiments, and comparisons are given in Section 6.

2. Multiscale Image Operators on Lattices

A poset is any set equipped with a partial ordering $\leq$. The supremum ($\bigvee$) and infimum ($\bigwedge$) of any subset of a poset is its lowest upper bound and greatest lower bound, respectively; both are unique if they exist. A poset is called a (sup-) inf-semillattice if the (supremum) infimum of any finite collection of its elements exists. A (sup-) inf-semilattice is called complete if the (supremum) infimum of arbitrary collections of its elements exist. A poset is called a (complete) lattice if it is simultaneously a (complete) sup- and an inf-semilattice. An operator ψ on a complete lattice is called: *increasing* if it preserves the partial ordering $[f \leq g \Longrightarrow \psi(f) \leq \psi(g)]$; *idempotent* if $\psi^2 = \psi$; *antiextensive* (*resp., extensive*) if $\psi(f) \leq f$ (resp., $f \leq \psi(f)$). An operator ε (resp., δ) on a complete inf-semilattice (resp., sup-semilattice) is called an *erosion* (resp., *dilation*) if it distributes over the infimum (resp., supremum) of any collection of lattice elements; namely $\delta(\bigvee_i f_i) = \bigvee_i \delta(f_i)$ and $\varepsilon(\bigwedge_i f_i) = \bigwedge_i \varepsilon(f_i)$. A lattice operator is called an *opening* (resp., *closing*) if it is increasing, idempotent, and antiextensive (resp., extensive). A *negation* is a bijective operator $\nu \neq \mathrm{id}$ such that both ν and ν^{-1} are either decreasing or increasing and $\nu^2 = \mathrm{id}$, where id is the identity. An operator ψ is called *self-dual* if it commutes with a negation ν.

In this paper, the image space is the collection of signals defined on a continuous or discrete domain $\mathbb{E}$ and assuming values in $\mathbb{V}$, where $\mathbb{E} = \mathbb{R}^m$ or $\mathbb{Z}^m$, $m = 1, 2, \ldots$, and $\mathbb{V} \subseteq \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. The value set $\mathbb{V}$ is equipped with some partial ordering that makes it a complete lattice or inf-semilattice. This lattice structure is inherited by the image space by extending the partial order of $\mathbb{V}$ to signals pointwise.

2.1. *Multiscale Operators on Complete Lattices*

Classical lattice-based morphology (Heijmans, 1994; Serra, 1988) uses as image space the *complete lattice*

$\mathcal{L}$ of signals $f : \mathbb{E} \to \mathbb{V}$ with values in $\mathbb{V} = \overline{\mathbb{R}}$ or $\overline{\mathbb{Z}}$. In $\mathcal{L}$ the signal ordering is defined by $f \leq g \Leftrightarrow f(x) \leq g(x), \forall x$, and the signal infimum and supremum are defined by $(\bigwedge_i f_i)(x) = \sup_i f_i(x)$ and $(\bigvee_i f_i)(x) = \inf_i f_i(x)$. Assume first $\mathbb{E} = \mathbb{R}^m$. Let $B = \{x : \|x\| \leq 1\}$ denote the unit-radius ball in $\mathbb{R}^m$ w.r.t. the Euclidean metric $\|\cdot\|$ and let $tB = \{tb : b \in B\}$ be its version at scale $t \geq 0$. The simplest multiscale dilation/erosion on $\mathcal{L}$ are the *Minkowski flat dilation/erosion* of an image f by tB:

$$\begin{aligned} \delta^t_B(f)(x) &\triangleq (f \oplus tB)(x) = \bigvee_{a \in tB} f(x-a) \\ \varepsilon^t_B(f)(x) &\triangleq (f \ominus tB)(x) = \bigwedge_{a \in tB} f(x+a) \end{aligned} \tag{2}$$

We shall also need the *multiscale conditional dilation and erosion* of a *marker* ('seed') image f given a *reference* ('mask') image r:

$$\begin{aligned} \delta_{tB}(f \mid r) &\triangleq (f \oplus tB) \wedge r \\ \varepsilon_{tB}(f \mid r) &\triangleq (f \ominus tB) \vee r \end{aligned} \tag{3}$$

Iterating the unit-scale conditional dilation (erosion) yields the *conditional reconstruction opening* (*closing*) of r from f:

$$\begin{aligned} \rho^-_B(f \mid r) &\triangleq \delta^\infty_B(f \mid r) = \bigvee_{n \geq 1} \delta^n_B(f \mid r) \\ \rho^+_B(f \mid r) &\triangleq \varepsilon^\infty_B(f \mid r) = \bigwedge_{n \geq 1} \varepsilon^n_B(f \mid r) \end{aligned} \tag{4}$$

where, for any operator ψ and any positive integer n, ψ^n denotes the n-fold composition of ψ with itself and $\psi^\infty = \lim_{n\to\infty} \psi^n$ if the limit[2] exists.

Another important pair is the geodesic dilation and erosion. First we define them for sets $X \subseteq \mathbb{E}$. Let $R \subseteq \mathbb{E}$ be a *reference* (mask) set and consider its *geodesic metric* $d_R(x, y)$ equal to the length of the geodesic path connecting the points x and y inside R. If $B_R(x,t) = \{p \in R : d_R(x,p) \leq t\}$ is the geodesic closed ball with center x and radius $t \geq 0$, then the multiscale geodesic set dilation of X given R is defined by

$$\Delta^t(X \mid R) \triangleq \{p \in R : B_R(p,t) \cap X \neq \emptyset\} \tag{5}$$

By using threshold decomposition and synthesis of an image f from its *upper level sets* $X_h(f) \triangleq \{x \in \mathbb{E} : f(x) \geq h\}$ we can synthesize a flat geodesic dilation for images by using as generator its set counterpart. Then, a possible definition of geodesic erosion is via negation. The resulting *multiscale geodesic dilation and erosion* of f given a reference image r are

$$\begin{aligned} \delta^t(f \mid r)(x) &\triangleq \sup\{h \leq r(x) : x \in \Delta^t(X_h(f) \mid X_h(r))\} \\ \varepsilon^t(f \mid r)(x) &\triangleq -\delta^t(-f \mid -r) \end{aligned} \tag{6}$$

The multiscale geodesic dilation and erosion possess a semigroup property:

$$\delta^t \delta^s = \delta^{t+s}, \quad \varepsilon^t \varepsilon^s = \varepsilon^{t+s}, \quad \forall s,t \geq 0 \tag{7}$$

whereas their conditional counterparts do not: $\delta_{tB}(\delta_{sB}(f \mid r) \mid r) \neq \delta_{(t+s)B}(f \mid r)$. By letting $t \to \infty$ the geodesic dilation (erosion) yields the *geodesic reconstruction opening* ρ^- *(closing* ρ^+*)* of r from f:

$$\begin{aligned} \rho^-(f \mid r) &\triangleq \delta^\infty(f \mid r) = \bigvee_{t \geq 0} \delta^t(f \mid r) \\ \rho^+(f \mid r) &\triangleq \varepsilon^\infty(f \mid r) = \bigwedge_{t \geq 0} \varepsilon^t(f \mid r) \end{aligned} \tag{8}$$

The above limit δ^∞ (ε^∞) can also be reached using iterations δ^n (ε^n) for $n \to \infty$ since, due to the semigroup property, the geodesic dilation (erosion) at integer scales $t = n$ can be obtained via n-fold iteration of the unit-scale operator.

The definitions of multiscale operators remain valid for signals defined on $\mathbb{Z}^m$ if we use integer scales $t = n$, replace the Euclidean metric with a discrete metric on $\mathbb{Z}^m$ and define the multiscale set nB as the n-fold Minkowski sum of B with itself. Further, the geodesic operators for sampled signals need a connectivity grid on $\mathbb{Z}^m$.

2.2. *Image Operators on Reference Semilattices*

In Keshet (2000), Heijmans and Keshet (2000, 2001) a recent approach for a self-dual morphology was developed based on inf-semilattices. Now, the image space is the collection of signals $f : \mathbb{E} \to \mathbb{V}$, where $\mathbb{V} = \mathbb{R}$ or $\mathbb{Z}$. The value set $\mathbb{V}$ becomes a *complete inf-semilattice* (*cisl*) if we select an arbitrary *reference* element $r \in \mathbb{V}$ and use the following partial ordering:

$$a \preceq_r b \Longleftrightarrow r \wedge b \leq r \wedge a \quad \text{and} \quad r \vee b \geq r \vee a \tag{9}$$

The corresponding infimum in $(\mathbb{V}, \preceq_r)$ is

$$\begin{aligned} \curlywedge_i a_i &\triangleq \left(r \wedge \bigvee_i a_i\right) \vee \bigwedge_i a_i \\ &= \left(r \vee \bigwedge_i a_i\right) \wedge \bigvee_i a_i \end{aligned} \tag{10}$$

The ordering $\preceq$ coincides with the *activity ordering* in Boolean lattices (Meyer and Serra, 1989; Heijmans, 1994).

Given a reference image $r(x)$, a valid signal cisl ordering is $f \preceq_r g$ defined as $f(x) \preceq_{r(x)} g(x)\ \forall x$. The corresponding signal cisl infimum becomes

$$\begin{aligned} \left(\curlywedge_i f_i\right)(x) &\triangleq \left[r(x) \wedge \bigvee_i f_i(x)\right] \vee \bigwedge_i f_i(x) \\ &= \operatorname{med}\left[r(x), \bigvee_i f_i(x), \bigwedge_i f_i(x)\right] \end{aligned} \tag{11}$$

where med$(\cdot)$ denotes the median. Under the above cisl infimum, the image space becomes a cisl denoted henceforth by $\mathcal{F}_r$. Varying the reference signal r yields cisl's that are all isomorphic to each other. Significant in this paper is the cisl $\mathcal{F}_0$ with $r(x) = 0$. An isomorphism between $\mathcal{F}_0$ and an arbitrary cisl $\mathcal{F}_r$ is the bijection $\xi(f) = f + r$. Thus, if ψ_0 is an operator on $\mathcal{F}_0$, then its corresponding operator on $\mathcal{F}_r$ is given by

$$\psi_r(f) = \xi\psi_0\xi^{-1}(f) = r + \psi_0(f - r) \tag{12}$$

If ψ_0 is an erosion on $\mathcal{F}_0$ that is translation-invariant (TI) and self-dual, then ψ_r is also a self-dual TI erosion on $\mathcal{F}_r$. *Note*: the infimum, translation operator and negation operator on $\mathcal{F}_0$ are different from those on $\mathcal{F}_r$. For example, if $\nu_0(f) = -f$ is the negation on $\mathcal{F}_0$, then self-duality of ψ_0 means $\psi_0\nu_0 = \nu_0\psi_0$, whereas self-duality on $\mathcal{F}_r$ means $\psi_r\nu_r = \nu_r\psi_r$ where $\nu_r(f) = 2r - f$.

The simplest multiscale TI self-dual erosion on the cisl $\mathcal{F}_0$ is the operator

$$\psi_0^t(f)(x) = \left[0 \wedge \bigvee_{a \in tB} f(x-a)\right] \vee \bigwedge_{a \in tB} f(x-a) \tag{13}$$

The corresponding multiscale TI self-dual erosion on the cisl $\mathcal{F}_r$ is

$$\begin{aligned} &\psi_r^t(f)(x) \\ &\quad = r(x) + \left(\left[0 \wedge \left(\bigvee_{a \in tB} f(x-a) - r(x-a)\right)\right]\right. \\ &\qquad \left.\vee \left(\bigwedge_{a \in tB} f(x-a) - r(x-a)\right)\right) \end{aligned} \tag{14}$$

3. Multiscale Triphase Operators and Levelings

3.1. Triphase Operators

Consider operators in the complete lattice $\mathcal{L}$. To define levelings as in Meyer (1998) and Matheron (1997) and to generalize them we shall begin by introducing the concept of triphase operators.

Definition 1. Given two increasing operators α_p and β_p on $\mathcal{L}$ such that α_p is antiextensive and β_p is extensive, a *parallel triphase* operator λ_p is defined by

$$\begin{aligned} \lambda_p(f \mid r; \alpha_p, \beta_p) &\triangleq \alpha_p(f) \vee (r \wedge \beta_p(f)) \\ &= \beta_p(f) \wedge (r \vee \alpha_p(f)) \end{aligned} \tag{15}$$

where r is the reference signal and f is a marker signal.

The subscript p in the above operators denotes 'parallel'. The prototypical example is when α_p and β_p are a flat erosion ε_B and dilation δ_B, respectively. Then, λ_p becomes a composition of a conditional erosion and a conditional dilation. Hence, we may call a general parallel triphase 'conditional'. In this paper we also define a more general type of triphase operators as follows.

Definition 2. Given two operators α_s and β_s from $\mathcal{L}^2$ to $\mathcal{L}$ that are increasing w.r.t. both arguments and, $\forall f, r$,

$$f \wedge r \leq \beta_s(f \mid r) \leq r \leq \alpha_s(f \mid r) \leq f \vee r, \tag{16}$$

two *serial triphase* operators λ_1 and λ_2 are defined by

$$\begin{aligned} \lambda_1(f \mid r; \alpha_s, \beta_s) &\triangleq \alpha_s(f \mid \beta_s(f \mid r)), \\ \lambda_2(f \mid r; \alpha_s, \beta_s) &\triangleq \beta_s(f \mid \alpha_s(f \mid r)). \end{aligned} \tag{17}$$

The subscript s in the above operators refers to 'serial'. Any parallel triphase operator becomes a serial one by setting $\alpha_s(f \mid r) = \alpha_p(f) \vee r$ and $\beta_s(f \mid r) = \beta_p(f) \wedge r$. However, the converse is not always true. Henceforth, we drop the subscripts s and p from

α, β, λ and focus on the serial case since it is more general. Thus, unless otherwise mentioned, by 'triphase operator' we shall mean a serial one. The triphase operators have four arguments: two signals f and r and two operators α and β. The signal arguments (f, r) are written as $(f \mid r)$ to emphasize their asymmetric roles (marker vs. reference) and to connect them later with conditional operators that use the same notation. If the operators α and β are known and fixed, we shall omit them and write $\lambda(f \mid r)$. If the reference r is assumed, we may write $\lambda(f)$, $\alpha(f)$, $\beta(f)$; if f is also implied, we may express the operator outputs simply as λ, α, β.

The inequality (16) and the assumption of being increasing immediately imply the following properties for the operators α, β.

Corollary 1.

(a) $f \leq r \Longrightarrow \alpha(f \mid r) = r$, *and*
$f \geq r \Longrightarrow \beta(f \mid r) = r$.
(b) $r \leq \alpha^2(f) \leq \alpha(f) \leq \alpha(f \vee r) \leq f \vee r$
(c) $f \wedge r \leq \beta(f \wedge r) \leq \beta(f) \leq \beta^2(f) \leq r$.

Lemma 1. *Let* $\lambda_1 = \lambda_1(f \mid r)$ *and* $\lambda_2 = \lambda_2(f \mid r)$ *be the two triphase operators of* (17). *Let* $\alpha = \alpha(f \mid r)$, $\beta = \beta(f \mid r)$. *Then for all* f, r:

(a) $\beta = \lambda_1 \wedge r \leq \lambda_1 \leq \lambda_1 \vee r \leq \alpha$.
(b) $\beta \leq \lambda_2 \wedge r \leq \lambda_2 \leq \lambda_2 \vee r = \alpha$.
(c) $\beta \leq \lambda_1 \leq \lambda_2 \leq \alpha$.

Proof: Lemma 1(a): From (16) it follows directly that $\beta \leq \lambda_1 \leq \alpha \leq f \vee \beta$. Taking the max and min of all terms in this inequality with r and using (16) yields that $\beta \leq \lambda_1 \wedge r \leq (f \vee \beta) \wedge r = \beta$ and $\lambda_1 \vee r \leq \alpha \vee r = \alpha$. These two inequalities yield (a). The proof of Lemma 1(b) is similar. Lemma 1(c): Apply (a) and (b) at each point x. If $f(x) \geq r(x)$, then $r(x) = \beta(x) \leq \lambda_1(x) \leq \lambda_2(x) = \alpha(x) \leq f(x)$. If $f(x) \leq r(x)$, then $f(x) \leq \beta(x) = \lambda_1(x) \leq \lambda_2(x) \leq \alpha(x) = r(x)$. Thus, in both cases $\lambda_1 \leq \lambda_2$. □

Assume now that the operators α and β *commute* in the definition of the serial triphase (17), and hence $\lambda_1 = \lambda_2 = \lambda$. Then, Lemma 1 and its proof directly imply the following simplified properties for the single common triphase λ.

Proposition 1. *Assume that* $\alpha(f \mid \beta(f \mid r) = \beta(f \mid \alpha(f \mid r))$ *and let* $\lambda = \lambda(f \mid r)$ *be the common serial triphase operator. Then for all* f, r:

$$f \wedge r \leq \beta = \lambda \wedge r \leq \lambda \leq \lambda \vee r = \alpha \leq f \vee r \tag{18}$$

Thus, at each point x,

$$\begin{aligned} f(x) \geq r(x) &\Longrightarrow r(x) = \beta(x) \leq \lambda(x) = \alpha(x) \leq f(x) \\ f(x) \leq r(x) &\Longrightarrow f(x) \leq \beta(x) = \lambda(x) \leq \alpha(x) = r(x) \end{aligned} \tag{19}$$

The case $\lambda_1 = \lambda_2$ is useful and necessary for many of the subsequent results. Two sufficient conditions for this are provided next.

Proposition 2. *The operators* α *and* β *commute in the Definition* (17) *of a serial triphase operator* λ, *and hence* $\lambda(f \mid r) = \alpha(f \mid \beta(f \mid r)) = \beta(f \mid \alpha(f \mid r))$, *if any of the following two conditions holds*:

(a) λ *is a parallel triphase operator as in* (15), *i.e., if* $\alpha(f \mid r) = \alpha_p(f) \vee r$ *and* $\beta(f \mid r) = \beta_p(f) \wedge r$, *where* α_p, β_p *are increasing and* $\alpha_p(f) \leq f \leq \beta_p(f)$.
(b) $\alpha(f \mid r) = \alpha(f \vee r \mid r)$ *and* $\beta(f \mid r) = \beta(f \wedge r \mid r)$ *for all* f, r.

Proof: Proposition 2(a): In the parallel case, $\alpha(f \mid \beta(f \mid r)) = \alpha_p(f) \vee [\beta_p(f) \wedge r]$ is equal to $[\alpha_p(f) \vee \beta_p(f)] \wedge [\alpha_p(f) \vee r]$ which in turn equals $\beta_p(f) \wedge [\alpha_p(f) \vee r] = \beta(f \mid \alpha(f \mid r))$. Proposition 2(b): Let $\lambda(f \mid r) = \alpha(f \mid \beta(f \mid r))$. Denote $S^+ = \{x : f(x) > r(x)\}$ and $S^- = \{x : f(x) < r(x)\}$. Then, $\beta(x) = \beta(f \mid r)(x) = r(x)$ for points $x \notin S^-$. Further, since $\beta(f) = \beta(f \wedge r)$, the value $\beta(x)$ at any point $x \in S^-$ is a β-dependent function of values $f(p)$ and $r(p)$ only for points p inside S^-. Now, by Lemma 1, $\lambda(x) = \alpha(f \mid \beta)(x) = \beta(x)$ for $x \in S^-$, whereas for $x \in S^+$ $\lambda(x)$ is an α-dependent function of values $f(q)$ and $\beta(q) = r(q)$ for points $q \in S^+$. Hence, $\lambda(x) = \alpha(x)$ when $f(x) \geq r(x)$ and $\lambda(x) = \beta(x)$ when $f(x) \leq r(x)$. If we commute α and β in the definition of λ, its values at each point will not change because they will still depend in the same way on the same values of α and β over the same regions S^+, S^-. □

A significant property becomes now evident for the subclass of serial triphase operators whose constituent operators α and β *commute* and hence yield a single triphase operator

$$\lambda(f \mid r) = \alpha(f \mid \beta(f \mid r)) = \beta(f \mid \alpha(f \mid r)) \tag{20}$$

The action of such an operator λ at points x where $f(x) > r(x)$ (resp., $f(x) < r(x)$) is determined only by α (resp., β). Also points x where $f(x) = r(x)$ are not affected by any of the α, β, λ. Specifically,

$$\lambda(f \mid r)(x) = \begin{cases} \beta(f \mid r)(x), & \text{if } f(x) < r(x) \\ \alpha(f \mid r)(x), & \text{if } f(x) < r(x) \\ r(x), & \text{if } f(x) = r(x) \end{cases} \tag{21}$$

Henceforth, in this paper we shall only deal with triphase operators that satisfy (20) and hence (21). Some general properties of triphase operators follow next.

Proposition 3. *Let $\lambda(f \mid r) = \alpha(f \mid \beta(f \mid r)) = \beta(f \mid \alpha(f \mid r))$ be a serial triphase operator. Then*:

(a) *λ is antiextensive in the cisl $\mathcal{F}_r$; i.e., $\lambda(f \mid r) \preceq_r f$.*
(b) *λ is increasing in $\mathcal{F}_r$; i.e., $f \preceq_r g \Longrightarrow \lambda(f \mid r) \preceq_r \lambda(g \mid r)$.*
(c) *Let (α_1, β_1) and (α_2, β_2) create two triphase operators $\lambda'(f) = \lambda(f \mid r; \alpha_1, \beta_1)$ and $\lambda''(f) = \lambda(f \mid r; \alpha_2, \beta_2)$, respectively. If $\alpha_1 \geq \alpha_2$ and $\beta_1 \leq \beta_2$, then $\lambda''(f) \preceq_r \lambda'(f)$, $\forall f$.*
(d) *If α and β are dual of each other, then λ is self-dual; i.e., if $\alpha(-f \mid -r) = -\beta(f \mid r)$, then $\lambda(-f \mid -r) = -\lambda(f \mid r)$.*

Proof: Proposition 3(a): By (18) we have $r \vee \lambda \leq r \vee f$ and $r \wedge \lambda \geq r \wedge f$. Hence (a) is true by the definition of $\preceq_r$. Proposition 3(b): $f \preceq_r g$ means that $f \wedge r \geq g \wedge r$ and $f \vee r \leq g \vee r$. This partitions all points x into three types: (i) $r(x) > f(x) \geq g(x)$, (ii) $r(x) < f(x) \leq g(x)$, (iii) $r(x) = f(x)$ and $g(x)$ is arbitrary. At points of type (iii) (b) holds because $\lambda(f)(x) = r(x)$. At points of the first (resp., second) type where $r > f \geq g$ (resp., $r < f \leq g$), λ preserves this ordering because its values are identical with those of β (resp., α) which is increasing. Proposition 3(c): Let $\alpha_i = \alpha_i(f \mid r)$ and $\beta_i = \beta_i(f \mid r)$, $i = 1, 2$. We need to show that (i) $r \vee \lambda'' \leq r \vee \lambda'$ and (ii) $r \wedge \lambda'' \geq r \wedge \lambda'$. By Proposition 1, at points x where $f(x) \leq r(x)$, $\lambda' = \beta_1 \leq \beta_2 = \lambda'' \leq r$; hence, $r \vee \lambda'' = r \vee \lambda'$. At points x where $f(x) \geq r(x)$, $\lambda' = \alpha_1 \geq \alpha_2 = \lambda'' \geq r$; hence, $r \vee \lambda'' \leq r \vee \lambda'$. Thus (i) holds at all points. Similarly we can prove (ii). Hence, (c) is true. Proposition 3(d) results directly from the definition of λ and the duality between α and β. □

3.2. Levelings

Meyer (1998) and Matheron (1997) defined as a leveling of r any signal f such that $\delta_B(f) \wedge r \leq f \leq \varepsilon_B(f) \vee r$. It can be shown that this is equivalent to $f = \lambda(f \mid r)$, where λ is the parallel (conditional) triphase formed by the flat erosion ε_B and dilation δ_B in place of α and β. By generalizing the triphase to be a serial one, we propose the following alternative definition of levelings.

Definition 3. A signal f is a called a parallel or serial *leveling* of r iff it is a fixed point of the parallel or serial, respectively, triphase operator, i.e. $f = \lambda(f \mid r)$.

The 'parallel' leveling may also be called 'conditional.' The following is a necessary and sufficient condition for f to be a λ-induced leveling of r:

$$f = \lambda(f \mid r) \Longleftrightarrow \beta(f \mid r) \leq f \leq \alpha(f \mid r) \tag{22}$$

Since any triphase operator λ is antiextensive, a leveling of a reference r from a marker f can possibly be obtained by *iterating* λ to infinity, or equivalently by taking the cisl infimum of all iterations of λ. Specifically, the limit

$$\begin{aligned} \Lambda(f \mid r) &\triangleq \lambda^\infty(f \mid r) \\ &= \curlywedge_{n \geq 1} \lambda^n(f) \preceq_r \cdots \preceq_r \lambda(f) \preceq_r f \end{aligned} \tag{23}$$

exists in the cisl $\mathcal{F}_r$. The map $r \mapsto \Lambda(f \mid r)$ is an increasing and antiextensive (in $\mathcal{F}_r$) operator. Next we analyze the iterations of λ and show that, if $\lambda\lambda^\infty = \lambda^\infty$, then Λ is also idempotent and its output is a leveling.

Proposition 4. *Let $\lambda(\cdot \mid \cdot; \alpha, \beta)$ be a serial triphase operator.*

(a) *$\alpha^n \downarrow \alpha^\infty = \bigwedge_{n \geq 1} \alpha^n$, $\beta^n \uparrow \beta^\infty = \bigvee_{n \geq 1} \beta^n$, and for all f, r*

$$\lambda^\infty(f \mid r) = \alpha^\infty(f \mid \beta^\infty(f \mid r)) \tag{24}$$

(b) *$\lambda\lambda^\infty = \lambda^\infty$ iff $\alpha\alpha^\infty = \alpha^\infty$ and $\beta\beta^\infty = \beta^\infty$.*
(c) *If α is a lattice erosion and β is a lattice dilation, then $\lambda\lambda^\infty = \lambda^\infty$.*
(d) *If $\lambda\lambda^\infty = \lambda^\infty$, then $\lambda^\infty(f \mid r)$ is a leveling of r for all f and $\lambda^\infty(\cdot \mid r)$ is idempotent.*

Proof: Proposition 4(a): By Corollary 1(b), α^n is a decreasing sequence bounded below by r; hence, as $n \to \infty$, the limit α^∞ exists and equals $\bigwedge_{n \geq 1} \alpha^n$. Similarly for β^n which is an increasing sequence upper bounded by r. Then, (24) follows from (21), since at points x where $f(x) \geq r(x)$ (resp. $f(x) \leq r(x)$) λ

and all its iterations coincide with the corresponding iterations of α (resp. β). Proposition 4(b): Assume first that $\alpha\alpha^\infty = \alpha^\infty$ and $\beta\beta^\infty = \beta^\infty$. Then, the pointwise representation (21) of λ yields $\lambda\lambda^\infty(f \mid r) = \alpha\alpha^\infty(f \mid \beta\beta^\infty(f \mid r)) = \lambda^\infty(f \mid r)$. The converse is proven similarly. Proposition 4(c): If α is an erosion, then $\alpha\alpha^\infty = \alpha(\bigwedge_{n\geq 1}\alpha^n) = \bigwedge_{n\geq 1}\alpha^{n+1} = \alpha^\infty$. Similarly, if β is a dilation, then $\beta\beta^\infty = \beta^\infty$. The rest follows from (b). Proposition 4(d): It was proven in (Heijmans, 1994, p. 452) for an arbitrary lattice operator ψ that, if $\psi^n \to \psi^\infty$ and $\psi\psi^\infty = \psi^\infty$, then ψ^∞ is idempotent and $\psi^\infty(f) = f$ iff $\psi(f) = f$. Hence, (d) is a direct application of this result. □

Henceforth, we shall deal only with triphase operators λ for which $\lambda\lambda^\infty = \lambda^\infty$. In such cases, the map $r \mapsto \Lambda(f \mid r)$ is called a *leveling operator* since its output is a leveling. Note that, Λ is an increasing, antiextensive and idempotent operator, and hence a *semilattice opening*, in the cisl $\mathcal{F}_r$. It is also an increasing and idempotent operator, and hence a *morphological filter*, in the complete lattice $\mathcal{L}$.

3.3. Semigroups of Multiscale Triphase Operators and Semilattice Erosions

Assume real-valued signals defined on $\mathbb{R}^m$. If we replace the operators α and β with the multiscale conditional flat erosion and dilation by B of (3) we obtain a parallel *multiscale conditional triphase* operator

$$\begin{aligned}\lambda_{tB}(f \mid r)(x) &\triangleq \varepsilon_{tB}(f)(x) \vee [r(x) \wedge \delta_{tB}(f)(x)]\\ &= \curlywedge_{a\in tB} f(x-a) \qquad (25)\end{aligned}$$

It is called 'conditional' because it can be written as a serial triphase operator involving conditional dilation and erosion:

$$\lambda_{tB}(f \mid r) = \varepsilon_{tB}(f \mid \delta_{tB}(f \mid r)) = \delta_{tB}(f \mid \varepsilon_{tB}(f \mid r)) \qquad (26)$$

By replacing the conditional dilation and erosion in (26) with their geodesic counterparts from (6) we obtain a serial *multiscale geodesic triphase* operator

$$\lambda^t(f \mid r) \triangleq \varepsilon^t(f \mid \delta^t(f \mid r)) = \delta^t(f \mid \varepsilon^t(f \mid r)) \qquad (27)$$

For both the conditional and the geodesic triphase, its constituent erosion and dilation operators satisfy the basic properties of the operators α and β required for the definitions of triphase operators. Further, they *commute*. In the conditional case, this is simple to see. The geodesic triphase is also well-defined since δ^t and ε^t commute. This follows from Proposition 2(b) because they satisfy the following additional property.

Proposition 5. *Let δ^t and ε^t be the multiscale geodesic dilation and erosion of* (6). *Then, for all f, r, t, $\delta^t(f \mid r) = \delta^t(f \wedge r \mid r)$ and $\varepsilon^t(f \mid r) = \varepsilon^t(f \vee r \mid r)$.*

Proof: For the geodesic set dilation of (5), we have $\Delta^t(X \mid R) = \Delta^t(X \cap R \mid R)$. The geodesic signal dilation δ^t of (6) is a flat operator generated from Δ^t via threshold superposition. Then, since $X_h(f) \cap X_h(r) = X_h(f \wedge r)$, where $X_h(f)$ are the upper level sets of f, we obtain $\delta^t(f \mid r) = \delta^t(f \wedge r \mid r)$. The dual result for the geodesic erosion follows via negation. □

Calling the above triphase operators 'multiscale' can be easily justified in the conditional case λ_{tB} where the scale parameter t coincides with the radius of the disk used for eroding or dilating the marker. In the geodesic case, the scale t equals the geodesic distance used to create the geodesic balls.

Comparing (25) with (14) reveals that $\lambda_{tB}(\cdot \mid r)$ becomes a multiscale *translation-invariant* (*TI*) semilattice erosion on $\mathcal{F}_r$ if r is constant. In particular, if $r = 0$, then λ_{tB} becomes a multiscale TI self-dual erosion on $\mathcal{F}_0$. For non-constant r, λ_{tB} is generally neither TI nor an erosion. In constrast, as proven next, the geodesic triphase is a semilattice erosion in $\mathcal{F}_r$, although *not* TI.

Proposition 6. *The geodesic triphase operator $\lambda^t(f \mid r) = \varepsilon^t(f \mid \delta^t(f \mid r)$ is a semilattice erosion in the cisl $\mathcal{F}_r$; i.e., $\lambda^t(\curlywedge_i f_i \mid r) = \curlywedge_i \lambda^t(f_i \mid r)$.*

Proof: We use the pointwise representation (21) of λ^t. First, note that, using the definition (11) for the cisl infimum $\curlywedge$, Proposition 5, the $\vee$-distributivity of δ^t and its extensivity yields

$$\begin{aligned}\delta^t\left(\curlywedge_i f_i\right) &= \delta^t\left(r \wedge \bigvee_i f_i\right) \vee \delta^t\left(\bigwedge_i f_i\right)\\ &= \delta^t\left(\bigvee_i f_i\right) \vee \delta^t\left(\bigwedge_i f_i\right)\\ &= \delta^t\left(\bigvee_i f_i\right) = \bigvee_i \delta^t(f_i). \qquad (28)\end{aligned}$$

Now, consider points x where $f(x) = \curlywedge_i f_i(x) \leq r(x)$ and hence $\lambda^t(f \mid r)(x) = \delta^t(f \mid r)(x)$. The condition $\curlywedge_i f_i(x) \leq r(x)$ is equivalent to $r(x) \wedge \bigvee_i f_i(x) \leq r(x)$ and $\bigwedge_i f_i(x) \leq r(x)$. This implies that $\bigvee_i \delta^t(f_i)(x) \leq r(x)$ and hence $\curlywedge_i \delta^t(f_i)(x) = \bigvee_i \delta^t(f_i)(x)$. This and (28) imply $\delta^t(\curlywedge_i f_i)(x) = \curlywedge_i \delta^t(f_i)(x)$. Similarly, at points x where $\curlywedge_i f_i(x) \geq r(x)$ we have $\lambda^t(f \mid r)(x) = \varepsilon^t(f \mid r)(x)$. By working as for the dilation we can show that, $\varepsilon^t(\curlywedge_i f_i) = \bigwedge_i \varepsilon^t(f_i)$. Further, the condition $\curlywedge_i f_i(x) \geq r(x)$ implies that $\curlywedge_i \varepsilon^t(f_i)(x) = \bigwedge_i \varepsilon^t(f_i)(x)$. Therefore, λ^t distributes over $\curlywedge$ since $\lambda^t(\curlywedge_i f_i)(x) = \curlywedge_i \lambda^t(f_i)(x)$ at all points. □

The geodesic triphase is the most important triphase operator because it obeys a semigroup. This will allow us later to find its PDE generator.

Proposition 7.

(a) *As $t \to \infty$, $\lambda^t(f \mid r)$ yields the geodesic leveling which is the composition of the geodesic reconstruction opening and closing*:

$$\Lambda(f \mid r) \triangleq \lambda^\infty(f \mid r) = \rho^-(f \mid \rho^+(f \mid r)) = \rho^+(f \mid \rho^-(f \mid r)) \tag{29}$$

(b) *The multiscale family $\{\lambda^t(\cdot \mid r) : t \geq 0\}$ forms an additive semigroup*:

$$\lambda^t(\lambda^s(\cdot \mid r) \mid r) = \lambda^{t+s}(\cdot \mid r), \quad \forall t, s \geq 0. \tag{30}$$

(c) *For a zero reference ($r = 0$), the multiscale geodesic triphase operator becomes identical to its conditional counterpart and the multiscale TI semilattice erosion*:

$$r = 0 \Longrightarrow \psi_0^t(f) = \lambda^t(f \mid 0) = \lambda_{tB}(f \mid 0) \tag{31}$$

(d) *For any r, the multiscale TI semilattice erosion $\psi_r^t(f) = r + \psi_0^t(f - r)$ obeys a semigroup*:

$$\psi_r^t \psi_r^s = \psi_r^{t+s} \quad \forall t, s \geq 0. \tag{32}$$

Proof: Proposition 7(a) results from the definitions of the geodesic triphase and reconstruction openings and from Proposition 4. Proposition 7(b): Let $\lambda^s = \lambda^s(f \mid r)$, $\delta^s = \delta^s(f \mid r)$ and $\varepsilon^s = \varepsilon^s(f \mid r)$. Then, by (21), at points x where $f(x) \leq r(x)$ we have $\lambda^s(x) = \delta^s(x)$. Hence, by (7), $\lambda^t \lambda^s = \delta^t \delta^s = \delta^{t+s} = \lambda^{t+s}$. Similarly, at points x where $f(x) \geq r(x)$ we have $\lambda^s(x) = \varepsilon^s(x)$ and hence $\lambda^t \lambda^s = \varepsilon^t \varepsilon^s = \varepsilon^{t+s} = \lambda^{t+s}$. Thus, at all points $\lambda^t \lambda^s$ acts equivalently to λ^{t+s}. Proposition 7(c): If $r = 0$, $\psi_0^t(f) = \lambda_{tB}(f \mid 0)$ from their definitions. Further, to show that $\lambda^t(f \mid 0) = \lambda_{tB}(f \mid 0)$ observe first that, by (21), (6) and (25), at points x where $f(x) \leq 0$:

$$\begin{aligned}\lambda^t(f \mid 0)(x) &= \delta^t(f \mid 0)(x) \\ &= \sup\{h \leq 0 : x \in \Delta^s(X_h(f) \mid \mathbb{E})\} \\ &= 0 \wedge \bigvee_{a \in tB} f(x-a) = \lambda_{tB}(f \mid 0)(x).\end{aligned}$$

Similarly it can be proven when $f(x) \geq 0$. Proposition 7(d): By writing $\psi_r^t = \xi \psi_0^t \xi^{-1}$ with $\xi(f) = f + r$ we obtain $\psi_r^t \psi_r^s = \xi \psi_0^{t+s} \xi^{-1} = \psi_r^{t+s}$ since $\psi_0^t(\cdot) = \lambda^t(\cdot \mid 0)$ obeys a semigroup. □

From the semigroup property (30), the n-fold iteration of the unit-scale geodesic triphase operator concides with its multiscale version at integer scale $t = n$. The same is true for the multiscale TI semilattice erosions. It is not generally true, however, for the conditional triphase operator $\lambda_B(f \mid r)$, which does *not* obey a semigroup. Further, its iterations converge to the *conditional leveling* $\Lambda_B(f \mid r) = \lambda_B^\infty(f \mid r)$ which is smaller w.r.t. $\preceq_r$ than the geodesic leveling $\Lambda(f \mid r) = \lambda^\infty(f \mid r)$ of (29). Namely, since $\delta_B^n(f \mid r) \geq \delta^n(f \mid r)$ and $\varepsilon_B^n(f \mid r) \leq \varepsilon^n(f \mid r)$, by Proposition 3

$$r \preceq_r \Lambda_B(f \mid r) \preceq_r \Lambda(f \mid r)\,. \tag{33}$$

4. PDEs for 1D Levelings and Semilattice Erosions

4.1. Leveling PDE

Consider a 1D reference image $r(x)$ and a marker image $f(x)$ on $\mathbb{R}$, both real and continuous. We start evolving the marker image by producing the *multiscale geodesic triphase evolutions*

$$u(x, t) = \lambda^t(f \mid r)(x) = \delta^t(f \mid \varepsilon^t(f \mid r))(x) \tag{34}$$

The initial value is $u(x, 0) = f(x)$. In the limit we obtain the final result $u(x, \infty)$ which will be the leveling $\Lambda(f \mid r)$. Attempting to find a generator PDE for the function u, we shall analyze the following evolution rule:

$$\frac{\partial u}{\partial t}(x,t) = \lim_{s \downarrow 0} \frac{u(x,t+s) - u(x,t)}{s} \tag{35}$$

By using the semigroup (30) that u satisfies and the pointwise representation (21) of triphase operators, the evolution rule becomes

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= \lim_{s \downarrow 0} \frac{\lambda^s(u(x,t) \mid r)(x) - u(x,t)}{s} \\ &= \begin{cases} \lim_{s\downarrow 0}[\delta^s(u(x,t) \mid r)(x) - u(x,t)]/s, \\ \quad \text{if } u(x,t) < r(x) \\ \lim_{s\downarrow 0}[\varepsilon^s(u(x,t) \mid r)(x) - u(x,t)]/s, \\ \quad \text{if } u(x,t) > r(x) \\ 0, \quad \text{if } u(x,t) = r(x) \end{cases} \end{aligned} \tag{36}$$

We shall show later that, at points where the partial derivatives exist, this rule becomes the following PDE: $u_t = -\text{sign}(u - r)|u_x|$. Starting from a continuous marker $f(x)$, the evolutions $u(x,t)$ remain continuous for all x, t. However, even if the initial image f is differentiable, at finite scales $t > 0$, the above triphase evolution may create shocks (i.e., discontinuities in the derivatives). One way to propagate these shocks (as done in solving evolution PDEs of the Hamilton-Jacobi type with level-set methods (Osher and Sethian, 1988)) is to use conservative monotone difference schemes that pick the correct weak solution satisfying the entropy condition. An alternative way we propose to deal with shocks is to replace the standard derivatives with morphological sup/inf derivatives. For example, let

$$\mathcal{M}_x u(x,t) \triangleq \lim_{s\downarrow 0} \left[\bigvee_{|a|\leq s} u(x+a,t) - u(x,t) \right] \Big/ s$$

be the *sup-derivative* of $u(x,t)$ along the x-direction, if the limit exists. If the one-sided right derivative $\partial_{+x}u(x,t)$ and left derivative $\partial_{-x}u(x,t)$ of u along the x-direction exist, then its sup-derivative also exists and is equal to

$$\mathcal{M}_x u(x,t) = \max[0, \partial_{+x}u(x,t), -\partial_{-x}u(x,t)] \tag{37}$$

The nonlinear derivative $\mathcal{M}$ leads next to a more general PDE that can handle discontinuities in $\partial u/\partial x$.

Theorem 1. *Let $u(x,t) = \lambda^t(f \mid r)(x)$ be the scale-space function of multiscale geodesic triphase operations with initial condition $u(x,0) = f(x)$. Assume that r is continuous and f is continuous with left and right derivatives at all x.*

(a) *If the partial sup-derivative $\mathcal{M}_x u$ exists at some (x,t), then*

$$\frac{\partial u}{\partial t}(x,t) = \begin{cases} \mathcal{M}_x(u)(x,t), & if\ u(x,t) < r(x) \\ -\mathcal{M}_x(-u)(x,t), & if\ u(x,t) > r(x) \\ 0, & if\ u(x,t) = r(x) \end{cases} \tag{38}$$

(b) *If the partial left and right derivatives $\partial_{\pm x}u$ exist at some (x,t), then*

$$\frac{\partial u}{\partial t}(x,t) = \begin{cases} \max[0, \partial_{+x}u(x,t), -\partial_{-x}u(x,t)], \\ \quad if\ u(x,t) < r(x) \\ \min[0, \partial_{+x}u(x,t), -\partial_{-x}u(x,t)], \\ \quad if\ u(x,t) > r(x) \\ 0, \quad if\ u(x,t) = r(x) \end{cases} \tag{39}$$

(c) *If the two-sided partial derivative $\partial u/\partial x$ exists at some (x,t), then u satisfies*

$$\frac{\partial u}{\partial t}(x,t) = -\text{sign}[u(x,t) - r(x)] \left| \frac{\partial u}{\partial x}(x,t) \right| \tag{40}$$

Proof: Theorem 1(a): In the case $u(x,t) < r(x)$, the numerator of the right-hand side of (36) equals

$$\sup\{h \leq r(x) : x \in \Delta^s[X_h(u) \mid X_h(r)]\} - u(x,t)$$

Due to the continuity of u and r, for s sufficiently small, $u(y,t) < r(y)$ for all $y \in [x-s, x+s]$ and the interval $[x-s, x+s]$ will lie inside $X_h(r)$ for $h \leq \bigvee_{|a|\leq s} u(x-a,t)$. Hence, in this infinitesimally small neighborhood the geodesic dilation will become unconditional flat dilation and (36) becomes

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= \lim_{s\downarrow 0} \frac{\bigvee_{|a|\leq s} u(x-a,t) - u(x,t)}{s} \\ &= \mathcal{M}_x u(x,t). \end{aligned}$$

Similarly we prove the case $r(x) < u(x,t)$. If $r(x) = u(x,t)$, $\lambda^s(f \mid r)(x) = r(x)\ \forall s \geq t$ and hence $u_t(x,t) = 0$. Theorem 1(b) results directly from (a) and (37). Theorem 1(c) results directly from (b) and the fact that, if left and right derivatives exist and are equal, then the sup-derivative becomes equal to the magnitude $|u_x(x,t)|$ of the standard derivative. □

Thus, assuming that $\partial u/\partial x$ exists and is continuous, the nonlinear PDE (40) can generate the multiscale evolution of the initial image $u(x, 0) = f(x)$ under the action of the geodesic triphase operator. However, even if f is differentiable, as the scale t increases, this evolution can create shocks. In such cases, the more general PDE (39) that uses morphological derivatives still holds and can propagate the shocks provided the equation evolves in such a way as to give solutions that are piecewise differentiable with left and right derivatives at each point.

The PDE (40) has a varying coefficient $-\text{sign}(u-r)$ with scale-space dependence which controls the instantaneous growth and stops it whenever $u = r$. (Of course, there is no growth also at extrema where $u_x = 0$.) The control mechanism is of a switching type: For each t, at points x where $u(x, t) < r(x)$ it acts as a dilation PDE and hence shifts parts of the graph of $u(x, t)$ with positive (negative) slope to the left (right) but does not move the extrema points. Wherever $u(x, t) > r(x)$ the PDE acts as an erosion PDE and reverses the direction of propagation. The final result $u(x, \infty)$ is a leveling of r w.r.t. f.

4.2. *PDE for 1D Multiscale TI Semilattice Erosions*

Consider now on the cisl $\mathcal{F}_0$ the *multiscale TI semilattice erosions* of a real 1D image $f(x)$ by 1D line segments $tB = [-t, t]$:

$$
\begin{aligned}
v(x, t) &= \psi_0^t(f)(x) \\
&= \left[0 \wedge \bigvee_{|a| \leq t} f(x-a) \right] \vee \bigwedge_{|a| \leq t} f(x-a)
\end{aligned}
\tag{41}
$$

Since $v(x, t)$ is the special case of the corresponding function $u(x, t)$ for multiscale geodesic triphase operations when $r = 0$, we can use the leveling PDE (40) to generate the evolutions $v(x, t)$:

$$
\begin{aligned}
\partial v/\partial t &= -\text{sign}(v)|\partial v/\partial x| \\
v(x, 0) &= f(x)
\end{aligned}
\tag{42}
$$

If $r(x)$ is not zero, we can generate multiscale TI semilattice erosions $\psi_r^t(f) = r + \psi_0^t(f - r)$ of f, defined explicitly in (14), by the following PDE system

$$
\begin{aligned}
&\partial v/\partial t = -\text{sign}(v)|v_x|, \quad v(x, 0) = f(x) - r(x) \\
&\psi_r^t(f)(x) = r(x) + v(x, t)
\end{aligned}
\tag{43}
$$

The 1D images f and r above are assumed to be continuous and possess left and right derivatives everywhere.

Note: If $f - r$ has some zero values and is non-constant, then as $t \to \infty$ we obtain a leveling identical to the reference r, because $\psi_0^\infty(f - r) = 0$ and hence $\psi_r^\infty(f) = r$.

4.3. *Discretization and Numerical Algorithms*

To find a numerical algorithm for solving the previous PDEs, let U_i^n be the approximation of $u(x, t)$ on a grid $(i\Delta x, n\Delta t))$. Similarly, define $R_i \triangleq r(i\Delta x)$ and $F_i \triangleq f(i\Delta x)$. Consider the forward and backward difference operators:

$$
\begin{aligned}
D_{+x}U_i^n &\triangleq \left(U_{i+1}^n - U_i^n\right)/\Delta x, \\
D_{-x}U_i^n &\triangleq \left(U_i^n - U_{i-1}^n\right)/\Delta x
\end{aligned}
\tag{44}
$$

To produce a shock-capturing and entropy-satisfying numerical method for solving the leveling PDE (40) we approximate the more general PDE (39) by replacing time derivatives with forward differences and left/right spatial derivatives with backward/forward differences. This yields the following algorithm:

$$
\begin{aligned}
U_i^{n+1} = U_i^n - \Delta t\big[&\left(P_i^n\right)^+ \max\left(0, D_{-x}U_i^n, -D_{+x}U_i^n\right) \\
&+ \left(P_i^n\right)^- \max\left(0, -D_{-x}U_i^n, D_{+x}U_i^n\right)\big]
\end{aligned}
\tag{45}
$$

where $P_i^n = \text{sign}(U_i^n - R_i)$, $q^+ = \max(0, q)$, and $q^- = \min(0, q)$. Further, to avoid spurious numerical oscillations around zerocrossings of $f - r$, at each iteration we enforce the sign consistency

$$
\text{sign}\left(U_i^n - R_i\right) = \text{sign}(F_i - R_i), \quad \forall n, i \tag{46}
$$

We iterate the above scheme for $n = 0, 1, 2, \ldots$, starting from the initial data $U_i^0 = F_i$. For *stability*, $(\Delta t/\Delta x) \leq 0.5$ is required.

The above scheme can be expressed as *iteration of a discrete triphase operator* Φ acting on the cisl $\mathcal{F}_R$ of 1D sampled real-valued signals with reference R:

$$
U_i^{n+1} = \Phi\left(U_i^n\right), \quad \Phi(F_i) \triangleq \alpha(F_i) \vee [R_i \wedge \beta(F_i)],
\tag{47}
$$

where the operators α, β are given by, for $\theta = \Delta t/\Delta x$,

$$
\begin{aligned}
\alpha(F_i) &= \min[F_i, \theta F_{i-1} + (1-\theta)F_i, \\
&\qquad \theta F_{i+1} + (1-\theta)F_i], \\
\beta(F_i) &= \max[F_i, \theta F_{i-1} + (1-\theta)F_i, \\
&\qquad \theta F_{i+1} + (1-\theta)F_i].
\end{aligned}
\tag{48}
$$

By using ideas from methods of solving PDEs corresponding to hyperbolic conservation laws (Osher and Sethian, 1988), we can easily show that this scheme is conservative and monotone increasing for $\theta = \Delta t/\Delta x \leq 1$. Hence, it satisfies the entropy condition.

There are also other possible approximation schemes such as the conservative and monotone scheme proposed in Osher and Rudin (1990) to solve the edge-sharpening PDE $u_t = -\text{sign}(u_{xx})|u_x|$:

$$\begin{aligned} U_i^{n+1} &= U_i^n - \Delta t \Big[\left(P_i^n\right)^+ \sqrt{\left(\left(D_{-x}U_i^n\right)^+\right)^2 + \left(\left(D_{+x}U_i^n\right)^-\right)^2} \\ &\quad + \left(P_i^n\right)^- \sqrt{\left(\left(D_{+x}U_i^n\right)^+\right)^2 + \left(\left(D_{-x}U_i^n\right)^-\right)^2} \Big] \end{aligned} \tag{49}$$

In order to solve the leveling PDE, we have modified this scheme to enforce the sign consistency condition (46). The combined algorithm can be expressed via the iteration of a discrete triphase operator Φ as in (47) but with different α and β:

$$\begin{aligned} &\alpha(F_i) \\ &\quad = F_i - \theta\sqrt{[\max(F_i - F_{i-1}, 0)]^2 + [\min(F_{i+1} - F_i, 0)]^2}, \\ &\beta(F_i) \\ &\quad = F_i + \theta\sqrt{[\min(F_i - F_{i-1}, 0)]^2 + [\max(F_{i+1} - F_i, 0)]^2} \end{aligned} \tag{50}$$

This second approximation scheme is more diffusive and requires more computation per iteration than the first scheme. Thus, as the main numerical algorithm to solve the leveling PDE, we henceforth adopt the first scheme (47),(48), which is based on discretizing the morphological derivatives. Examples of running this algorithm are shown in Figs. 1 and 3. An important question is whether the two above algorithms converge. The answer is affirmative as proved next.

Proposition 8. *If* $\Phi(\cdot) = \alpha(\cdot) \vee [R \wedge \beta(\cdot)]$ *and* (α, β) *are either as in* (48) *or as in* (50), *then* Φ *is a parallel triphase operator and the sequence* $U^{n+1} = \Phi(U^n)$, $U^0 = F$, *converges to a unique limit* $U^\infty = \Phi^\infty(F)$. *For digital images* F, R *assuming a finite number of gray levels, the limit* $\Phi^\infty(F)$ *is a conditional leveling of* R *from* F.

Proof: In both cases (48) and (50) the increasing operators α and β are antiextensive and extensive, respectively, w.r.t. the standard lattice ordering $\leq$. Hence, Φ is a valid parallel triphase operator, which can be written as a serial one $\Phi(F \mid R) = \alpha_s(F \mid \beta_s(F \mid R)$ by setting $\alpha_s(F \mid R) = R \vee \alpha(F)$ and $\beta_s(F \mid R) = R \wedge \beta(F)$. In the cisl $\mathcal{F}_R$, Φ is an antiextensive operator w.r.t. the cisl ordering $\preceq_R$. Hence, the limit $\Phi^\infty(F)$ is the cisl infimum of $\Phi^n(F)$ which exists and is unique. For digital images with a finite number of gray levels, the iteration of α_s, β_s will converge in a finite number of iterations, say $\alpha_s^{n+1} = \alpha_s^n$ and $\beta_s^{m+1} = \beta_s^m$ for some integers n, m. This implies that $\alpha_s\alpha_s^\infty = \alpha_s^\infty = \alpha_s^n$ and $\beta_s\beta_s^\infty = \beta_s^\infty = \beta_s^m$. Hence, by Proposition 4, $\Phi\Phi^\infty = \Phi^\infty$ and Φ^∞ creates a leveling. □

If $\Delta t = \Delta x$, then the α and β operators (48) of the discrete triphase operator Φ in (47) become erosion and dilation, respectively, by a unit-scale window $B = \{-1, 0, 1\}$. Further, the corresponding PDE numerical algorithm coincides with the iterative discrete algorithm of Meyer (1998) for constructing levelings.

5. PDEs for 2D Levelings and Semilattice Erosions

A straighforward extension of the leveling PDE from 1D to 2D images results by replacing the term $-|u_x|$ creating 1D multiscale erosions with the term $-\|\nabla u\|$ generating multiscale erosions by disks. Then the 2D leveling PDE becomes:

$$\begin{aligned} \partial u(x, y, t) &= -\,\text{sign}[u(x, y, t) - r(x, y)]\|\nabla u\| \\ u(x, y, 0) &= f(x, y) \end{aligned} \tag{51}$$

As in the 1D case, $u(x, y, t) = \lambda^t(f \mid r)(x, y)$ is a scale-space function holding the 2D multiscale geodesic triphase evolutions of the marker image $f(x, y)$ within the reference image $r(x, y)$. Of course, we could select any other PDE modeling the intermediate growth kernel by shapes other than the disk, but the disk has the advantage of creating an isotropic growth.

For discretization, let $U_{i,j}^n$ be the approximation of $u(x, y, t)$ on a computational grid $(i\Delta x, j\Delta y, n\Delta t)$ and set the initial condition $U_{ij}^0 = F_{ij} = f(i\Delta x, j\Delta y)$. Then, by replacing the magnitudes of standard derivatives with morphological derivatives and by expressing the latter with left and right derivatives which are approximated with backward and forward differences,

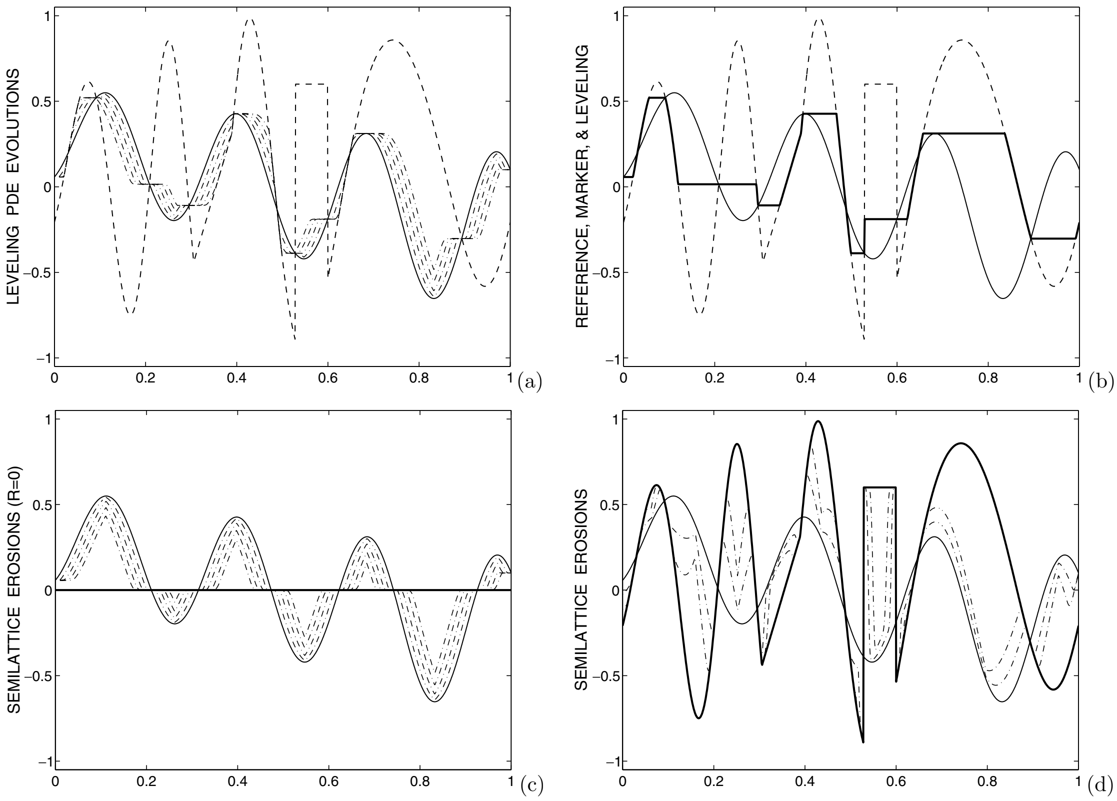


Figure 3. Multiscale triphase evolutions of 1D signals generated by PDEs. (a) A reference signal $r(x)$ shown (dash line), a marker signal $f(x)$ (thin solid line) and its evolutions $u(x,t)$ (dashdot line) generated by the leveling PDE $u_t = -\mathrm{sign}(u-r)|u_x|$, $u(x,0) = f(x)$, at times $t = n25\Delta t$, $n = 1, 2, 3, 4$. (b) The reference r, the marker f, and the leveling $u(x,\infty)$ (thick solid line). (c) Multiscale TI semilattice erosions $v(x,t)$ of $f(x)$ w.r.t. a zero reference generated by the PDE $v_t = -\mathrm{sign}(v)|v_x|$, $v(x,0) = f(x)$, at $t = n25\Delta t$, $n = 1, 2, 3, 4$. (d) Multiscale TI semilattice erosions $v(x,t)+r(x)$ of $f(x)$ w.r.t. the non-constant reference $r(x)$, generated by the PDE $v_t = -\mathrm{sign}(v)|v_x|$, $v(x,0) = f(x)-r(x)$, at $t = n25\Delta t$, $n = 1, 2$. ($\Delta x = 0.001$, $\Delta t = 0.0005$.)

we arrive at the following entropy-satisfying scheme for solving the 2D leveling PDE (51):

$$\begin{aligned} U_{i,j}^{n+1} &= \Phi\big(U_{i,j}^{n}\big), \quad \Phi(F_{ij}) \triangleq [R_{ij} \wedge \beta(F_{ij})] \vee \alpha(F_{ij}), \\ \alpha(F_{ij}) &= F_{ij} - \Delta t \sqrt{\max{}^2[0, D_{-x}F_{ij}, -D_{+x}F_{ij}] + \max{}^2[0, D_{-y}F_{ij}, -D_{+y}F_{ij}]} \\ \beta(F_{ij}) &= F_{ij} + \Delta t \sqrt{\max{}^2[0, -D_{-x}F_{ij}, D_{+x}F_{ij}] + \max{}^2[0, -D_{-y}F_{ij}, D_{+y}F_{ij}]} \end{aligned} \tag{52}$$

For stability, $(\Delta t/\Delta x + \Delta t/\Delta y) \leq 0.5$ is required. As in the 1D case, this scheme converges to a discrete conditional leveling. Examples of running the above 2D algorithm are shown in Fig. 4. In all image experiments based on PDEs we used $\Delta x = \Delta y = 1$, $\Delta t = 0.25$ as space-time steps.

As a by-product of the 2D leveling PDE, the multiscale TI semilattice erosions (14) of a marker image f by disks B w.r.t. a reference image r can be generated as follows:

$$\begin{aligned} \partial v/\partial t &= -\mathrm{sign}(v)\|\nabla v\|, \\ v(x,y,0) &= f(x,y) - r(x,y), \\ \psi_r^t(f)(x,y) &= r(x,y) + v(x,y,t) \end{aligned} \tag{53}$$

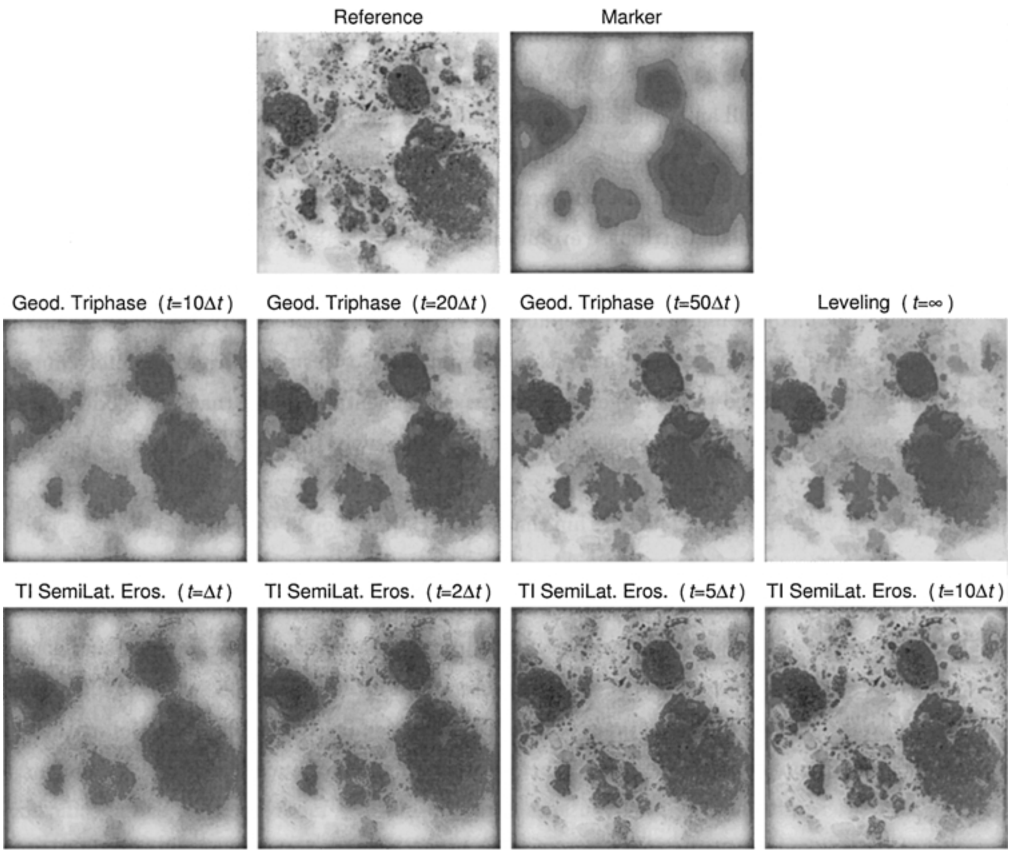


Figure 4. Multiscale triphase evolutions and levelings of soilsection images generated by PDEs. The marker image $f(x, y)$ was obtained from a convolution of the reference $r(x, y)$ with a 2D Gaussian of $\sigma = 8$. Second row: triphase evolutions (geodesic semilattice erosions) $u(x, y, t)$ generated by the leveling PDE (51). Third row: multiscale TI semilattice erosions $v(x, y, t)$ generated by the PDE (53).

6. Discussion

We conclude by providing some insights on the behavior of levelings and multiscale semilattice erosions via several image experiments. Then, we also comment on the advantages of PDE-based algorithms for generating these lattice scale-spaces.

As shown in Figs. 4, 5, and 6, the leveling limit is strongly dominated by the structure of the reference image. Although the selection of markers suitable for producing levelings with various designable properties is still an open issue, it appears that a smooth version of the reference works well as a marker for applications of image simplification and segmentation. In Fig. 5 we experimented with a binary edge map as reference whereas the marker was a smooth version of the same original image. Here the intermediate triphase evolutions (geodesic semilattice erosions) toward the leveling seemed useful for adding image region details back to the edge map. Finally, as shown in Figs. 5 and 6, the intermediate multiscale TI semilattice erosions seem potentially applicable to mixing or morphing the marker image into the reference, even if the two images are completely unrelated. On comparing the speed of convergence, we have experimentally found that the geodesic triphase evolutions toward levelings converge to the limit more slowly than the multiscale TI semilattice erosions.

The basic *algebraic discrete* algorithm that Meyer (1998) developed to construct levelings of digital images is the iteration of the conditional triphase operator $\lambda(F_i) = \varepsilon_B(F_i) \vee [R_i \wedge \delta_B(F_i)]$, where δ_B and ε_B are flat dilation and erosion by a discrete unit-scale disk-like set B. Now we know that this converges to a discrete conditional leveling $\Lambda_{algdiscr}$. If levelings can be modeled and generated by such an algebraic discrete model, why then use PDEs for levelings and semilattice erosions? In addition to the well-known advantages

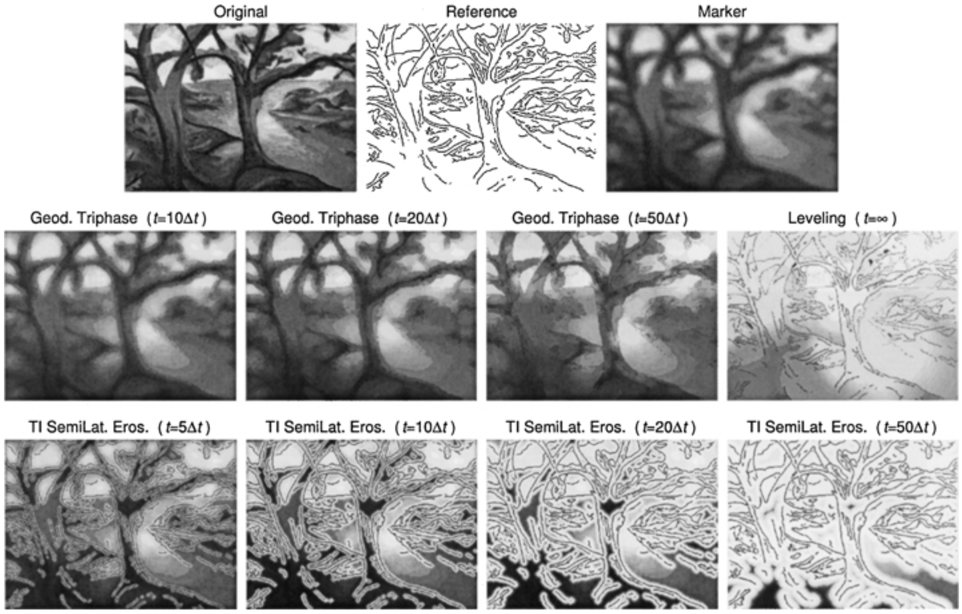


Figure 5. First Row: the reference image resulted from applying the Canny edge detector to the original image, and the marker image is a Gaussian convolution of the original. Second row: multiscale geodesic triphase evolutions converging to a leveling. Third row: Multiscale TI semilattice erosions. (All evolutions were generated by PDEs.)

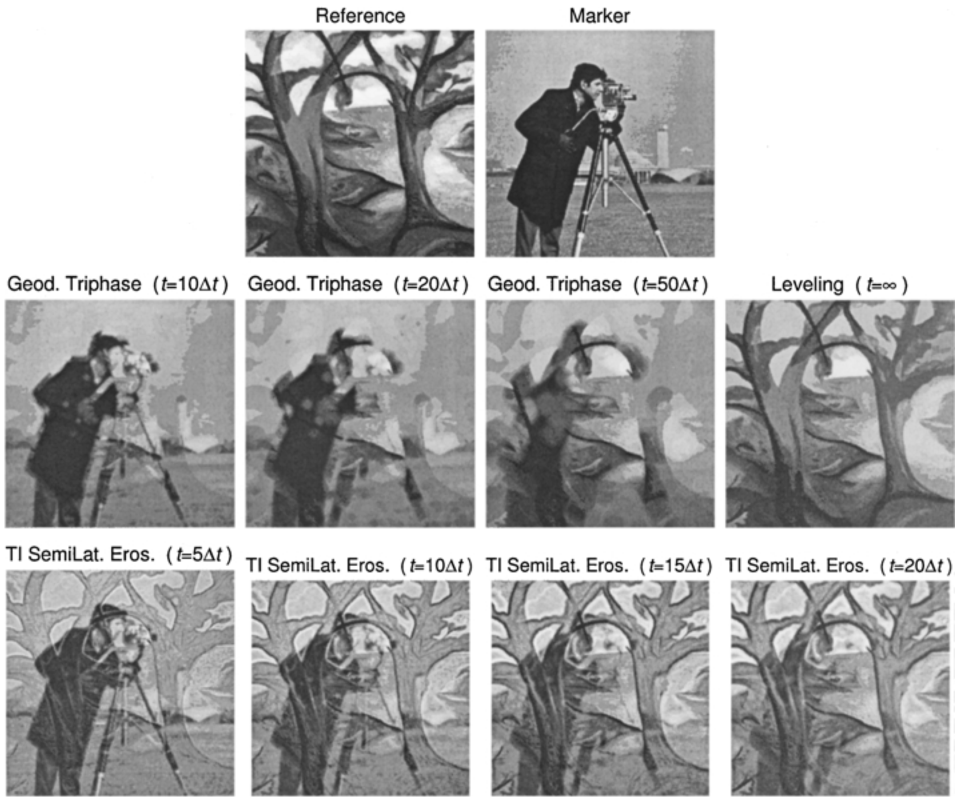


Figure 6. Multiscale geodesic triphase evolutions converging to a leveling and multiscale TI semilattice erosions of a marker image w.r.t. an unrelated reference. (All evolutions were generated by PDEs.)

of the PDE approach (such as more insightful mathematical modeling, more connections with physics, better approximation of Euclidean geometry, and subpixel accuracy), there are also some advantages over the discrete modeling that are specific for the operators examined in this paper. The new PDE-based numerical algorithm (47),(48) converges to another discrete conditional leveling Λ_{pdenum}. If Λ_{true} is the sampled true (geodesic) leveling, then

$$r \preceq_r \Lambda_{algdiscr} \preceq_r \Lambda_{pdenum} \preceq_r \Lambda_{true}.$$

Hence, the algebraic discrete algorithm yields a result that has a larger absolute deviation[3] from the true solution than the PDE numerical algorithm. Further, the algebraic discrete algorithm is a special case of the PDE algorithm using the value $\theta = \Delta t/\Delta x = 1$, which makes it *unstable* (amplifies small errors).

In the 2D case we have an additional comparison issue: Although for the triphase evolutions toward levelings the desired result in segmentation applications is mainly the final limit, there may be other applications, for instance such as mixing/morphing images, where we need to stop the marker growth before convergence. For example, the mixing/morphing in Figs. 5 and 6 indicates that the multiscale TI semilattice operators can generate quite interesting intermediate results. In such cases as evolutions of 2D multiscale (geodesic or TI) semilattice erosions, the isotropy of the partially grown marker offered by the PDE approach is an advantage over the discrete algebraic approach.

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Notes

1. Notation often used for PDEs: $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$, $\nabla u = (u_x, u_y)$.
2. A sequence (f_n) in the lattice $\mathcal{L}$ is defined in [Heijmans (1994), chap.13] to *order converge* to a limit f, written as $f_n \to f$, if $\liminf f_n = \limsup f_n = f$, where liminf and limsup are defined using only $\bigvee$ and $\bigwedge$. The simplest case is monotone convergence: $f_n \downarrow f$ means that (f_n) is decreasing and $f = \bigwedge_n f_n$, whereas $f_n \uparrow f$ means that (f_n) is increasing and $f = \bigvee_n f_n$. A sequence (ψ_n) of operators on $\mathcal{L}$ is defined to order converge to ψ, written as $\psi_n \to \psi$, if $\psi_n(f) \to \psi(f)$ for any $f \in \mathcal{L}$.
3. $f \preceq_r g$ implies $|f(x) - r(x)| \leq |g(x) - r(x)|\ \forall x$.

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